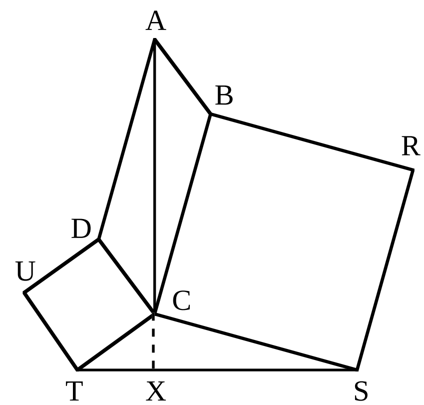
WISCONSIN MATHEMATICS SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET IV (1999-2000)

1. Let $\square$ be a binary operation defined on the integers, so that if x and y are any two integers, then $x \square y$ is some integer determined by x and y. Is it possible that $x \square (y \square z)$ is *never* equal to $(x \square y) \square z$ for any integers x, y, z? In other words, can a binary operation be "completely nonassociative"?

SOLUTION. Yes, it is possible to have a completely nonassociative binary operation on the integers. One such example is the operation defined by the formula $x \square y = y + 1$ for all integers x and y. To see this, we note that $x \square (y \square z) = (y \square z) + 1 = (z + 1) + 1 = z + 2$. On the other hand, $(x \square y) \square z = z + 1$, and this is never equal to $z + 2$.

2. Squares $BRSC$ and $DCTU$ are constructed on two sides of the parallelogram $ABCD$, as shown, and line segments $\overline{AC}$ and $\overline{ST}$ are drawn. Prove that $AC = ST$ and that $\overline{AC}$ (extended) is perpendicular to $\overline{ST}$.



SOLUTION. Observe that $AD = BC$ because opposite sides of a parallelogram are equal, and $BC = CS$ because all sides of a square are equal. Thus $AD = CS$, and clearly $DC = TC$. We see that $\angle TCS + \angle DCB + 2 \cdot 90^\circ = 360^\circ$, and thus $\angle TCS + \angle DCB = 180^\circ$. But also $\angle ADC + \angle DCB = 180^\circ$ because $\overline{AD}$ and $\overline{BC}$ are parallel. It follows that $\angle ADC = \angle TCS$. We can now conclude that $\triangle ADC \cong \triangle SCT$ by the SAS criterion, and hence $AC = TS$, as desired.

Now let X be the point where the extension of $\overline{AC}$ meets $\overline{TS}$. We see that $\angle ACD + 90^\circ + \angle TCX = 180^\circ$. Also, $\angle ACD = \angle CTX$ because these are corresponding angles in congruent triangles, and thus $\angle CTX + 90^\circ + \angle TCX = 180^\circ$. Since the three angles of $\triangle CTX$ also total 180°, it follows that $\angle CXT = 90^\circ$, and this completes the proof.

3. Suppose we are given a finite number of points on a circle, with each point labeled either 0 or 1. Then we can double the number of points by marking new points, one at each midpoint of the arc formed by two adjacent old points. Furthermore, if the two old points on either side of a new point have the same label, then the new point is labeled 0; otherwise it is labeled 1. Now start with two diametrically opposite points on the circle, one labeled 0 and the other 1, and repeatedly apply the doubling process as above. Prove that at every stage after the first, the total number of 0s is always equal to the number of 1s at the previous stage.

SOLUTION. First, we argue that at no stage can two points labeled 0 be neighbors. If this is wrong, then there is some first time when there are two neighboring 0s, and we see that one of those points (call it X) was just created, and the other (call it Y) is old. Since the newly created point X is labeled 0, it must lie between two old points with identical labels. But Y is one of those two old points, and Y is labeled 0, and hence the old point on the other side of X (call it Z) must also be labeled 0. At the previous stage, therefore, Y and Z are neighboring points which are both labeled 0, and this contradicts our assumption as to the first time a pair of neighboring 0s can occur.

Let us write O_n and Z_n to denote the number of 1s and 0s, respectively, at stage n. Since each neighbor of each 0 is a 1, we see that when we apply our doubling process, all new points that neighbor old 0s will be labeled 1, and all other new points are necessarily betweeen two old 1s and hence will be labeled 0. It follows that the number of new 1s is exactly twice the number of old 0s, since each old 0 gives rise to a 1 on either side, and hence $O_{n+1} - O_n = 2Z_n$. But there are exactly 2^n points at stage n, and thus $Z_n = 2^n - O_n$. Substituting this into the previous equation yields $O_{n+1} - O_n = 2(2^n - O_n)$, and by simplifying, we see that $O_{n+1} = 2^{n+1} - O_n$. Thus $O_n = 2^{n+1} - O_{n+1}$, and this is equal to Z_{n+1}. In other words, the number of 0s at stage $n+1$ is equal to the number of 1s at stage n. This is what we wanted to show.

4. Let S be a set of 51 positive integers, each of which is at most 100. Show that S contains a number that is a multiple of some other number in S.

SOLUTION. If m is an odd positive integer not exceeding 100, let A_m be the set all of those numbers not exceeding 100 that can be written in the form $2^e \cdot m$, where $e \geq 0$. For example, $A_1 = \{1, 2, 4, 8, 16, 32, 64\}$, $A_3 = \{3, 6, 12, 24, 48, 96\}$, $A_5 = \{5, 10, 20, 40, 80\}$ and $A_7 = \{7, 14, 28, 56\}$. (Note that if $m \geq 51$, then m is the only member of A_m.)

Since there are exactly 50 odd positive integers not exceeding 100, we have constructed 50 sets of the form A_m. If n is any integer in the range $1 \leq n \leq 100$, it is always possible to write $n = 2^e \cdot m$ for some odd integer m. (Starting with n, simply keep dividing by 2 until an odd integer is obtained.) It follows that every number from 1 to 100 inclusive lies in (at least) one of the 50 sets A_m.

Since our given set S has 51 members, there must be two of them (say u and v, with $u < v$) that lie in the same set A_m. It follows that $u = 2^e \cdot m$ and $v = 2^f \cdot m$ with $e < f$, and thus we see that v/u is an integer. (In fact, v/u is a power of 2.) Thus v is a multiple of u, as required.

5. Let p be a prime number and suppose that $\frac{1}{p} = \frac{1}{a} + \frac{1}{b}$, where a and b are positive integers. Find all possibilities for p if one of a or b is a perfect square.

SOLUTION. If we clear denominators in the given equation, we obtain $ab = pb + pa$, and thus ab is a multiple of p. Since p is prime, it follows that either a or b (or both) is a multiple of p and so it is no loss to assume that a is a multiple of p, and we can write $a = pu$.

We now have $pub = pb + p^2u$, and thus $ub = b + pu$. Thus $pu = b(u-1)$ and we see that pu is a multiple of $u - 1$. But no prime divisor of $u - 1$ divides u, and so it follows that $u - 1$ must divide p, and thus $u - 1$ is either 1 or p. If $u - 1 = 1$ then $u = 2$ and $b = 2p = a$. In this case, the only way that a or b can be a square is if $p = 2$.

The remaining possibility is that $u - 1 = p$, in which case $b = u = p + 1$ and $a = pu = p(p+1)$. In this situation, a is definitely not a square since it is divisible by p to the first power only. If b is a square, write $b = r^2$ and observe that $p = b - 1 = r^2 - 1 = (r+1)(r-1)$. Since p is prime, the only possibility is that $r - 1 = 1$ and $r + 1 = p$, and in this case, we see that $r = 2$ and $p = 3$.