WISCONSIN MATHEMATICS SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET III (1999-2000)

1. Let a, b and c be nonnegative real numbers. Prove that

$$(a+b+c)^3 \geq a^3+b^3+c^3+24abc\,.$$

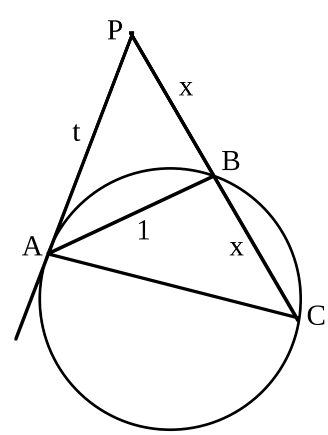
SOLUTION. We start with the inequality $a(b-c)^2+b(c-a)^2+c(a-b)^2 \geq 0$, which is valid since we are given that a, b and c are all nonnegative, and since the squares are nonnegative. If we expand the parentheses and rearrange the terms, we get $a^2b+a^2c+b^2a+b^2c+c^2a+c^2b \geq 6abc$. Next, we expand $(a+b+c)^3$ to obtain

$$(a+b+c)^3 = a^3+b^3+c^3+3(a^2b+a^2c+b^2a+b^2c+c^2a+c^2b)+6abc\,.$$

Finally, substituting our previous inequality yields

$$(a+b+c)^3 \geq a^3+b^3+c^3+3(6abc)+6abc = a^3+b^3+c^3+24abc\,.$$

2. Triangle ABC is inscribed in a circle, and side $AB=1$. The tangent to the circle at A meets the secant line through B and C at point P, and it happens that B is the midpoint of $\overline{PC}$. Compute the length of side AC.



SOLUTION. Recall that the angle between chord $\overline{AB}$ and tangent $\overline{AP}$ is equal in degrees to half of the arc determined by the chord. Also, inscribed $\angle ACB$ is equal in degrees to half of this arc, and so we deduce that $\angle ACB = \angle BAP$. Of course, we also have $\angle P = \angle P$, and we conclude that $\triangle ABP \sim \triangle CAP$ by the "angle-angle" similarity criterion. It follows that

$$\frac{AC}{AB} = \frac{AP}{BP} = \frac{CP}{AP}\,.$$

Write $AP=t$ and $BP=x=BC$. Since $AB=1$, we have $AC=t/x=2x/t$. From this, we see that $t^2=2x^2$, and thus $t^2/x^2=2$. Finally, we obtain $AC=t/x=\sqrt{2}$.

3. (New Year's Problem) I wish to write the number 1 as a sum of reciprocals of 2000 different positive integers. Is this possible? Prove that your answer is correct.

SOLUTION. Let us say that a positive number N is "acceptable" if 1 can be written as a sum of reciprocals of N different positive integers. For example, since $1=\frac{1}{2}+\frac{1}{3}+\frac{1}{6}$, we see that 3 is an acceptable number, and our problem is to decide whether or not 2000 is acceptable.

Suppose that we know that a certain number N is acceptable. Then, by definition, we can write $1=\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_N}$, where the numbers a_i are different positive integers. (Note also that none of the a_i can be equal to 1.) We now have

$$1=\frac{1}{2}+\frac{1}{2}=\frac{1}{2}+\frac{1}{2}\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_N}\right)=\frac{1}{2}+\frac{1}{2a_1}+\frac{1}{2a_2}+\cdots+\frac{1}{2a_N}\,,$$

and we have written 1 as a sum of reciprocals of $N+1$ positive integers 2, $2a_1$, $2a_2$ and so on, up to $2a_N$. Furthermore, these $N+1$ integers are all different since the a_i are all different and since none of them is equal to 1, so that none of the numbers $2a_i$ is equal to 2. We have now shown that if we know that N is acceptable, then so is $N+1$. Since we know that 3 is acceptable, it follows that 4 is acceptable, and thus 5 is acceptable, and hence 6 is acceptable, and so on. We conclude that all integers ≥ 3 are acceptable, and in particular, 2000 is acceptable.

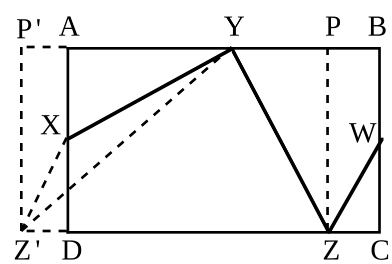
4. Let p be an odd prime number and write $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{p-1}$ as a fraction $\frac{a}{b}$ in reduced form, so that a and b are relatively prime positive integers. Prove that p divides a.

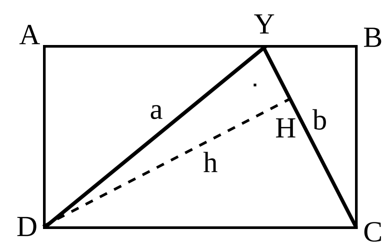
SOLUTION. Since p is odd, $p-1$ is even and we can pair off the summands so that

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{p-1}=\left(\frac{1}{1}+\frac{1}{p-1}\right)+\left(\frac{1}{2}+\frac{1}{p-2}\right)+\cdots+\left(\frac{1}{n}+\frac{1}{p-n}\right)$$

where $n=\frac{1}{2}(p-1)$ and $p-n=\frac{1}{2}(p+1)$. Since $\frac{1}{i}+\frac{1}{p-i}=\frac{p}{i(p-i)}$, we see that each pair adds to a fraction having p in the numerator and numbers not divisible by p in the denominator. When we combine these fractions, the sum becomes a'/b' where p divides a' and p does not divide the common denominator b'. In particular, even after deleting common factors in the numerator a' and denominator b' to obtain a/b in lowest terms, we see that the factor p of a' cannot be cancelled and consequently p divides a.

5. Suppose that the zigzag line $\overline{XYZW}$ has total length at most 2 units, and note that it is inscribed in the rectangle $ABCD$, as indicated. If $XD=WC$, show that the area of the rectangle $ABCD$ is at most 1 square unit.





SOLUTION. Draw the line $\overline{PZ}$ parallel to $\overline{BC}$, and then copy the rectangle $PBCZ$ to $P'ADZ'$, as indicated. Note that rectangles $ABCD$ and $P'PZZ'$ have the same area. Furthermore, since $XD=WC$, the line $\overline{Z'X}$ is parallel to and has the same length as $\overline{ZW}$. Thus the zigzag line $\overline{Z'XYZ}$ has the same total length as does $\overline{XYZW}$, and by assumption this is at most 2 units. Next, replace the line segments $\overline{Z'X}$ and $\overline{XY}$ by $\overline{Z'Y}$. Since $Z'X+XY\geq Z'Y$, we see that the zigzag line $\overline{Z'YZ}$ has total length at most 2 units, and this zigzag line is inscribed in the rectangle $P'PZZ'$. Since it suffices to prove that $P'PZZ'$ has area at most 1 square unit, we have reduced the original problem to the simpler situation of the second diagram where $X=D$ and $Z=W=C$. Here we have $a+b=DY+YC\leq 2$ and the goal is to show that the area of the rectangle $ABCD$ is at most equal to 1.

Now let line segment $\overline{DH}$ be an altitude in $\triangle DYC$ of length h. We note that the area of $ABCD$ is $DC\times DA$, while the area of $\triangle DYC$ is $\frac{1}{2}(DC\times DA)=\frac{1}{2}(DH\times YC)$. In particular, the area of $ABCD$ is twice that of $\triangle DYC$ and hence equal to hb. Thus, since h is clearly smaller than a, we see that $ABCD$ has area at most ab. But $ab=\frac{1}{4}[(a+b)^2-(a-b)^2]\leq\frac{1}{4}(a+b)^2\leq\frac{1}{4}2^2=1$, so we conclude that $ABCD$ has area at most 1 square unit.