WISCONSIN MATHEMATICS SCIENCE & ENGINEERING TALENT SEARCH

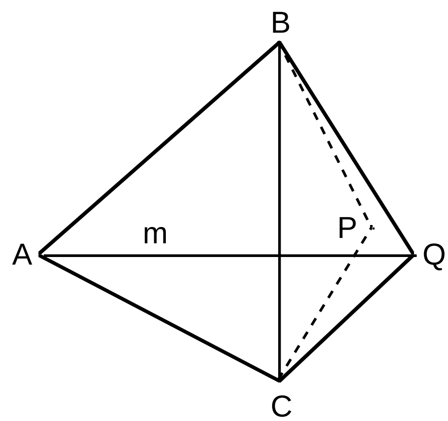
SOLUTIONS TO PROBLEM SET II (1999-2000)

1. Let S be a set of 51 different positive integers, each of which is at most 100. Show that there are two numbers in this set that differ by exactly 50.

SOLUTION. For integers i with $1 \le i \le 50$, let A_i denote the two-element set $\{i, 50+i\}$. Note that every positive integer $m \le 100$ is in one of these 50 sets since if $m \le 50$, we see that m is in A_m and if $m > 50$, then m lies in A_{m-50}.

The 51 members of the given set S are scattered among the 50 two-element sets, and thus at least one of these small sets must contain two members of S. If it is A_m that contains two members of S, we see that both m and $50+m$ lie in S, and these two members of S have a difference of 50.

2. Suppose that $\triangle ABC$ has acute angles, and that P is a point on the opposite side of line $\overline{BC}$ from A, as shown. If $PB^2 + AC^2 = PC^2 + AB^2$, prove that the line $\overline{AP}$ is perpendicular to $\overline{BC}$.



SOLUTION. Draw the line m through A perpendicular to $\overline{BC}$, as shown, and note that our goal is to prove that line m goes through point P. We can certainly assume that line m meets at least one of the line segments $\overline{PB}$ or $\overline{PC}$, and by renaming the points, if necessary, we can assume that m meets $\overline{PC}$. It follows that there is a point Q on line m, as indicated on the diagram, such that $PC = QC$.

Now consider the quadrilateral $ACQB$, and notice that its diagonals $\overline{BC}$ and $\overline{QA}$ are perpendicular. As we saw in the solution to Problem 2 of Problem Set I, the sum of the squares of the opposites sides of such a quadrilateral are equal, and thus $QB^2 + AC^2 = QC^2 + AB^2$. But $QC = PC$, and we are given that $PB^2 + AC^2 = PC^2 + AB^2$, so it follows from these equations that $PB = QB$. In other words, both point P and point Q lie on the circle of radius BP centered at B and also on the circle of radius CP centered at C. But P and Q lie on the same side of the line $\overline{BC}$ joining the centers of these two circles, and this is impossible if P and Q are different points. It follows that $P = Q$, and thus P lies on line m, as required.

3. Find all positive integers N such that the quantity

$$s = \sqrt[3]{2+\sqrt{N}} + \sqrt[3]{2-\sqrt{N}}$$

is a positive integer. Prove that your answer is correct.

SOLUTION. Write $x = \sqrt[3]{2+\sqrt{N}}$ and $y = \sqrt[3]{2-\sqrt{N}}$, so that $s = x+y$ and we have $s^3 = (x+y)^3 = x^3+y^3+3xy(x+y) = x^3+y^3+3xys$. Now $x^3 = 2+\sqrt{N}$ and $y^3 = 2-\sqrt{N}$, and thus $x^3+y^3 = 4$. Also, $xy = \sqrt[3]{(2+\sqrt{N})(2-\sqrt{N})} = \sqrt[3]{4-N}$. Therefore we have $s^3 = 4+3s\sqrt[3]{4-N}$, and hence $(s^3-4)/3s = \sqrt[3]{4-N}$.

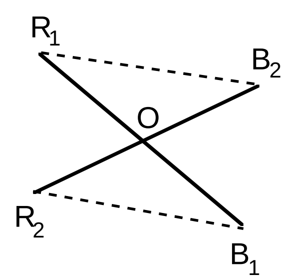
Since we require that N be a positive integer, the quantity on the right of the last equation is at most $\sqrt[3]{4} < 2$. Thus $(s^3 - 4)/3s < 2$, and this yields $s^3 - 6s < 4$. But $s^3 - 6s = s(s^2 - 6)$, and therefore the only positive integers s for which $(s^3 - 4)/3s < 2$ are $s = 1$ and $s = 2$. If $s = 2$, we have $\sqrt[3]{4 - N} = (s^3 - 4)/3s = 2/3$, and this is impossible since $2/3$ is clearly not the cube root of any integer.

We are left with the case $s = 1$, which yields $\sqrt[3]{4 - N} = (s^3 - 4)/3s = -1$, and thus $N = 5$ is the only possible solution. If we take $N = 5$, we must show that the value of s (which is the sum of the two cube roots in the statement of the problem) is actually equal to 1. We know that $(s^3 - 4)/3s = -1$, so $s^3 + 3s - 4 = 0$, and we must investigate the solutions of the polynomial equation $x^3 + 3x - 4 = 0$. Since we know that $x = 1$ is one solution, it follows that $x - 1$ must be a factor of $x^3 + 3x - 4$, and indeed we find that $x^3 + 3x - 4 = (x - 1)(x^2 + x + 4)$. If $s \neq 1$, we see that s must be a solution of the quadratic equation $x^2 + x + 4 = 0$. But by the quadratic formula, this equation has no real solutions. We conclude that $s = 1$ is the only possibility, and thus $N = 5$ is the only positive integer for which s is a positive integer.

4. Show that a product of four consecutive (positive) integers cannot be a perfect square.

SOLUTION. Suppose $m^2 = n(n + 1)(n + 2)(n + 3)$. Then $m^2 = [n(n + 3)][(n + 1)(n + 2)]$. The first of these factors is $n^2 + 3n$ and the second is $n^2 + 3n + 2$. Write $N = n^2 + 3n$. Then $m^2 = N(N + 2)$, and this lies strictly between N^2 and $(N + 1)^2$, a contradiction.

5. Given 20 points on a plane, 10 colored red and 10 blue; assume that no three are colinear. By a *connection* we mean a set of 10 line segments each one joining a different red point to a different blue one. How many connections are there? Now suppose we choose a connection with the property that the sum of the lengths of its 10 line segments is as small as possible. Prove that no two of its distinct line segments can intersect.



SOLUTION. The first red point can be connected to any of 10 blue points by a line segment. Once this blue point is chosen, the second red point can be connected to any of the remaining 9 blue points. Once these two blue points are chosen, the third red point can be connected to any of the remaining 8 blue points by a line segment, and so on. Thus the total number of connections is $10{\cdot}9{\cdot}8{\cdot}\cdots{\cdot}1 = 10!$.

Now suppose that we have a connection in which two of the line segments, say $\overline{R_1B_1}$ and $\overline{R_2B_2}$ intersect. Here, of course, R_1 and R_2 are distinct red points, while B_1 and B_2 are distinct blue points. Since no three of the 20 points are colinear, we must have the picture, as indicated above, with precisely one intersection point O. We replace these two line segments with $\overline{R_1B_2}$ and $\overline{R_2B_1}$ to obtain a new connection. Since the sum of the lengths of two sides of a triangle is larger than the third, we have $OR_1 + OB_2 > R_1B_2$ and $OB_1 + OR_2 > R_2B_1$. Adding these two equations yields $R_1B_1 + R_2B_2 > R_1B_2 + R_2B_1$. In other words, the sum of the lengths of the 10 line segments of the new connection is strictly smaller than the sum for the old connection. Consequently, if a connection is chosen with the property that the sum of the lengths of its 10 line segments is as small as possible, then its distinct line segments cannot intersect.