1. Let $S$ be a set of 51 different positive integers, each of which is at most 100. Show that there are two members of $S$ whose sum is 101.

2. In the figure, the diagonals $AC$ and $BD$ of the quadrilateral $ABCD$ are perpendicular. Given that $AB = 8$, $BC = 7$ and $DA = 4$, find $CD$, and prove that your answer is correct.

3. If $n$ is any positive integer, write $\sqrt{n} = m + r$, where $m$ is an integer and $0 \leq r < 1$. Of course, $m$ and $r$ depend upon $n$. For example, if $n = 20$, then $\sqrt{n} = 4.4721\ldots$, so $m = 4$ and $r = 0.4721\ldots$. Show that if $n$ is a multiple of $m$, then either $n$ is a square, $n$ is 1 less than a square, or $n$ is the product of two consecutive integers.

4. A number of high school students attend a dance. During the evening, each girl danced with at least one boy, but no boy danced with all the girls. Prove that there are two boys $B_1$ and $B_2$, and two girls $G_1$ and $G_2$, such that $B_1$ danced with $G_1$, $B_2$ danced with $G_2$, but $B_1$ did not dance with $G_2$, and $B_2$ did not dance with $G_1$.

5. Let $n$ be a positive integer. Prove that the product of all odd numbers from 1 to $4n - 1$ (inclusive) can never exceed $(4n^2 - 1)^n$.