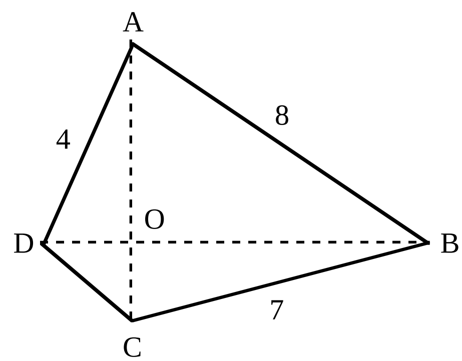
WISCONSIN MATHEMATICS SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET I (1999-2000)

1. Let S be a set of 51 different positive integers, each of which is at most 100. Show that there are two members of S whose sum is 101.

SOLUTION. Let T be the set of all numbers of the form $101 - s$, where s is in S. We see that T is also a set of 51 positive integers, each of which is at most 100. It follows that the sets S and T must overlap, since otherwise their union would consist of 102 numbers, all of which would lie in the set $\{1, 2, 3, \ldots, 100\}$, and this is clearly impossible. Suppose then, that x lies in S and also in T. Since x is in T, we see that $x = 101 - y$ for some member y of S, and thus $x + y = 101$. Note that $x \neq y$ since 101 is odd.

2. In the figure, the diagonals AC and BD of the quadrilateral $ABCD$ are perpendicular. Given that $AB = 8$, $BC = 7$ and $DA = 4$, find CD, and prove that your answer is correct.



SOLUTION. Let O be the point of intersection of the two diagonals. Then the four triangles $\triangle OAB$, $\triangle OBC$, $\triangle OCD$ and $\triangle ODA$ all have right angles at the vertex O. Thus, four applications of the Pythagorean Theorem yield

$$(AB)^2 = (OA)^2 + (OB)^2$$
$$(BC)^2 = (OB)^2 + (OC)^2$$
$$(CD)^2 = (OC)^2 + (OD)^2$$
$$(DA)^2 = (OD)^2 + (OA)^2.$$

Consequently,

$$(AB)^2 + (CD)^2 = (OA)^2 + (OB)^2 + (OC)^2 + (OD)^2 = (BC)^2 + (DA)^2.$$

Since $AB = 8$, $BC = 7$ and $DA = 4$, we conclude that

$$(CD)^2 = (BC)^2 + (DA)^2 - (AB)^2 = 7^2 + 4^2 - 8^2 = 1$$

and therefore $CD = 1$.

3. If n is any positive integer, write $\sqrt{n} = m + r$, where m is an integer and $0 \le r < 1$. Of course, m and r depend upon n. For example, if $n = 20$, then $\sqrt{n} = 4.4721\ldots$, so $m = 4$ and $r = 0.4721\ldots$. Show that if n is a multiple of m, then either n is a square, n is 1 less than a square, or n is the product of two consecutive integers.

SOLUTION. Since $\sqrt{n} = m + r$ and $0 \le r < 1$, we have $m \le \sqrt{n} < m + 1$ and squaring yields $m^2 \le n < (m+1)^2 = m^2 + 2m + 1$. Thus $m^2 \le n \le m^2 + 2m$ and $m \le n/m \le m + 2$. Now, we are given that m divides n, so n/m is an integer between m and $m+2$. The three possibilities are then: (1) $n/m = m$ so $n = m^2$ is a perfect square; (2) $n/m = m+1$ so $n = m(m+1)$ is the product of two consecutive integers; or (3) $n/m = m+2$ so $n = m^2 + 2m = (m+1)^2 - 1$ is 1 less than a square.

4. A number of high school students attend a dance. During the evening, each girl danced with at least one boy, but no boy danced with all the girls. Prove that there are two boys B_1 and B_2, and two girls G_1 and G_2, such that B_1 danced with G_1, B_2 danced with G_2, but B_1 did not dance with G_2, and B_2 did not dance with G_1.

SOLUTION. Choose B_1 to be a boy who danced with the largest number of girls. By assumption, B_1 did not dance with all the girls, so let G_2 be a girl who did not dance with B_1. Again, by assumption, G_2 danced with at least one boy, so choose a boy B_2 who danced with G_2. Note that B_1 and B_2 are different since B_2 danced with G_2, but B_1 did not. Finally, if B_2 danced with all the girls who danced with B_1, then since B_2 also danced with G_2, it would follow that B_2 danced with more girls than did B_1, and this contradicts our choice of B_1. Thus there must be at least one girl G_1 who danced with B_1 but not with B_2. It is now clear that the four students B_1, B_2, G_1 and G_2 satisfy our requirements.

5. Let n be a positive integer. Prove that the product of all odd numbers from 1 to $4n - 1$ (inclusive) can never exceed $(4n^2 - 1)^n$.

SOLUTION. The set $S = \{1, 3, 5, \ldots, 4n-1\}$ has exactly $2n$ members, of which n are less than $2n$ and n are greater than $2n$. It follows that every number in this set has the form $2n - k$ or $2n + k$ where k is an odd number with $1 \le k \le 2n - 1$.

We can multiply all the numbers in the set S by first multiplying together each of the pairs $\{2n-k, 2n+k\}$. Since $(2n-k)(2n+k) = 4n^2 - k^2 \le 4n^2 - 1$, we see that each of our pairs has product at most $4n^2 - 1$. Since there are exactly n such pairs, we conclude that the product of all the members of S is at most $(4n^2 - 1)^n$, as required.