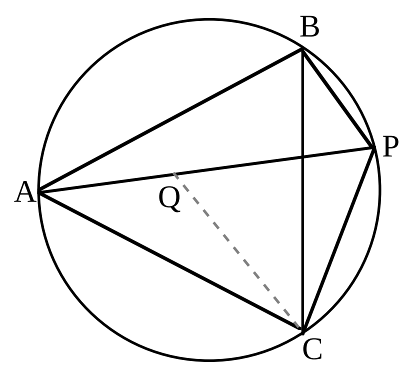
WISCONSIN MATHEMATICS SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET V (1998-99)

1. For some real number a, the polynomial $x^4 - 10x^2 + a = 0$ has four real roots that happen to be equally spaced on the real number line. Find the four roots and prove that there are no other possibilities.

SOLUTION. If r is a root of the fourth degree equation $x^4 - 10x^2 + a = 0$, then it is easy to see that $-r$ is also a root. Thus the four roots are symmetrically placed around the origin on the real line. Since the four roots are equally spaced, we see that if u is the smallest positive root, then the four roots are $-3u$, $-u$, u and $3u$. The two squares of these roots, namely u^2 and $9u^2$ must therefore be the roots of the quadratic equation $y^2 - 10y + a = 0$, where we have substituted y for x^2 in the original equation. It follows that $u^2 + 9u^2$, which is the sum of the two roots of this quadratic equation, must be 10. Thus $u^2 = 1$, and we have $u = 1$. The roots of the original equation are therefore -3, -1, 1 and 3, and there are no other possibilities. If we plug $x = 1$ into the fourth degree equation, we see that $a = 9$.

2. In the figure, equilateral $\triangle ABC$ is inscribed in a circle and a point P is chosen on the circle as shown, between vertices B and C. Show that $PA = PB + PC$.



SOLUTION. Draw the line $\overline{CQ}$ as shown, where Q lies on $\overline{AP}$ and $\angle ACQ = \angle BCP$. Then

$$\angle QCP = \angle QCB + \angle BCP$$
$$= \angle QCB + \angle ACQ = \angle ACB = 60^\circ .$$

Also, $\angle QPC = \angle ABC = 60^\circ$ since these angles both subtend the same arc of the circle. It follows that $\triangle QPC$ is equiangular, and hence it is equilateral. Thus $PC = PQ$ and, since $PA = PQ + QA$, it suffices to show that $QA = PB$.

Now consider $\triangle AQC$ and $\triangle BPC$. We have $AC = BC$ since $\triangle ABC$ is equilateral, and $QC = PC$ since $\triangle QPC$ is equilateral. Also, $\angle ACQ = \angle BCP$ by construction, and hence we have $\triangle AQC \cong \triangle BPC$ by SAS. It follows that $PB = QA$, as required, since these are corresponding parts of congruent triangles.

3. Let $\mathcal{T}$ be the set of all triples (a, b, c) of nonnegative integers. If $T = (a, b, c)$ is any member of $\mathcal{T}$, we write T^* to denote the new member of $\mathcal{T}$ obtained by $T^* = (|b-c|, |c-a|, |a-b|)$. In other words, the operation * replaces each entry in a triple by the difference of the other two entries. For example, we have $(2, 3, 4)^* = (1, 2, 1)$ and $(1, 2, 1)^* = (1, 0, 1)$. Prove that no matter what member of $\mathcal{T}$ we start with, if we keep applying our operation *, we will eventually obtain a triple that contains the number 0.

SOLUTION. Starting with a triple T, consider all the "descendants" of T, by which we mean the triples that result by applying our operation * repeatedly to T. Each descendant triple has some maximum entry, which is a nonnegative integer. Since every set of nonnegative integers contains a smallest member, there must be some smallest maximal entry m that occurs among all descendants of T. Let (u, v, w) be a descendant of T that has this number m as its maximum entry. Then each of u, v and w is a number between 0 and m. The largest of $|u - v|$, $|v - w|$ and $|w - u|$ is the maximum entry of $(u, v, w)^*$, and this cannot be smaller than m since $(u, v, w)^*$ is a descendant of T. But the only way that a difference of two numbers between 0 and m can be as large as m is if one of the numbers is 0. It follows that one of u, v or w is zero, and we are done.

4. Suppose that the prime number p can be written as a difference of two positive integer cubes. Show that when $4p$ is divided by 3 and the remainder is discarded, the result is the square of an odd integer.

SOLUTION. Write $p = a^3 - b^3$, and factor this to obtain $p = (a - b)(a^2 + ab + b^2)$. Since a and b are positive integers, we see that the second factor exceeds 1, and thus since p is prime, we must have $a - b = 1$. Now $a = b + 1$, and hence $p = (b + 1)^3 - b^3 = 3b^2 + 3b + 1$. It follows that $4p = 12b^2 + 12b + 4 = 3(4b^2 + 4b + 1) + 1$. Therefore, if we divide $4p$ by 3, the quotient is $4b^2 + 4b + 1$, with a remainder of 1. But $4b^2 + 4b + 1 = (2b + 1)^2$, and this is the square of an odd number, so the result is proved.

5. Ten pennies are placed on a table with "heads" side up, and they are arranged in a circle. A "move" consists of simultaneously turning over three adjacent pennies. Is it possible to carry out a sequence of moves that results in the pennies all being "tails" side up, and if so, what is the smallest possible number of moves needed to accomplish this?

SOLUTION. There are ten different ways to select three adjacent pennies in the circle, and so there are ten different possible moves. If we do one of each, then each penny will be flipped exactly three times, and since three is odd, each penny will end up tails. We will prove that this outcome cannot be achieved with fewer than ten moves.

To this end, let $\mathcal{S}$ be a sequence of n moves that results in all tails, where n is as small as possible. Each penny is flipped an odd number of times by the sequence $\mathcal{S}$, and we will show that in fact, each penny is flipped the *same* number of times. Otherwise, there will be two adjacent pennies (call them 1 and 2) that are flipped different odd numbers of times by $\mathcal{S}$. We can assume, therefore, that penny 2 was flipped at least two more times than penny 1. But there is only one move (flipping 2, 3 and 4) that flips penny 2 without also flipping penny 1, and so this move must have been done at least twice in $\mathcal{S}$.

Consider the new and shorter sequence of moves obtained by deleting two copies of the $(2, 3, 4)$-flip from $\mathcal{S}$. This deletion obviously has no effect on any of the seven pennies other than 2, 3 and 4, while each of 2, 3 and 4 is flipped exactly two fewer times by the shortened move sequence. Since two flips have no net effect, the final state of all ten pennies is unaffected by the deletion, and hence the shorter sequence of $n - 2$ moves also accomplishes the all-tails goal. This contradicts the minimality of n, and we have proved that each of the ten coins is flipped an equal number of times.

Finally, the n moves of $\mathcal{S}$ cause a total of $3n$ coin flips, and since each of the ten coins is flipped equally often, it follows that $3n$ must be a multiple of 10. The smallest positive integer n such that $3n$ is a multiple of 10 is clearly $n = 10$, and this shows that we cannot attain the all-tails condition in fewer than ten moves.