WISCONSIN MATHEMATICS SCIENCE & ENGINEERING TALENT SEARCH

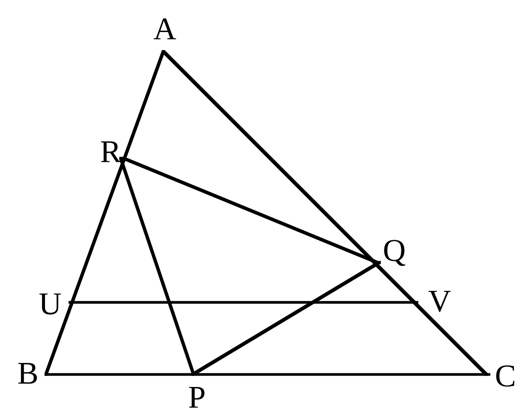
SOLUTIONS TO PROBLEM SET IV (1998-99)

1. Let a, b and c be real numbers with $a > 0$, and such that $a + b + c = 3$ and $abc = 1$. Find all possible values for a and prove that your answer is correct.

SOLUTION. Since $c = 3 - a - b$ and $abc = 1$, we can write $ab(3 - a - b) = 1$, or equivalently, $ab^2 + (a^2 - 3a)b + 1 = 0$. Thus, b is a root of the quadratic equation $ax^2 + (a^2 - 3a)x + 1 = 0$, and since this equation has the real root b, its discriminant cannot be negative. Recall that the discriminant d of a quadratic equation is the quantity that appears under the radical in the quadratic formula, and therefore, in our case, $d = (a^2 - 3a)^2 - 4a = a(a^3 - 6a^2 + 9a - 4)$. We can further simplify this expression by noting that $a^3 - 6a^2 + 9a - 4$ becomes 0 when $a = 1$. This tells us that $a - 1$ is also a factor of d, and by dividing, we obtain $d = a(a - 1)(a^2 - 5a + 4)$, and finally, $d = a(a - 1)^2(a - 4)$. We therefore have $a(a - 1)^2(a - 4) \geq 0$. If $a \neq 1$, then $a(a - 1)^2 > 0$, and consequently, $a - 4 \geq 0$ and $a \geq 4$. Thus, the only possibilities for a are $a = 1$ and $a \geq 4$.

We show next that if $a = 1$ or $a \geq 4$, then it is indeed possible to choose real numbers b and c so that $abc = 1$ and $a + b + c = 3$. In this case, the discriminant of our quadratic equation is nonnegative, and thus the equation has some real root b. We then have $ab(3 - a - b) = 1$, and so we can take $c = 3 - a - b$. (In fact, it turns out that c is the other root of the quadratic equation.)

2. In the figure, points P, Q and R are chosen on the sides of $\triangle ABC$ so that P lies $1/3$ of the way from B to C, Q lies $1/3$ of the way from C to A, and R lies $1/3$ of the way from A to B. Then points X and Y on the sides of $\triangle PQR$ are chosen so that X lies $1/3$ of the way from P to R and Y lies $1/3$ of the way from Q to P. Prove that line $\overline{XY}$ is parallel to $\overline{BC}$.



SOLUTION. Let U be the point on side $\overline{AB}$ such that $BU = (2/9)BA$, and similarly, let V the point on side $\overline{AC}$ such that $CV = (2/9)CA$. Since points U and V divide sides $\overline{AB}$ and $\overline{AC}$ of $\triangle ABC$ proportionally, it follows that $\overline{UV}$ is parallel to $\overline{BC}$. The proof will be complete, therefore, if we can show that points X and Y lie on $\overline{UV}$.

For this, note that $BU/BR = (BU/BA)(BA/BR) = (2/9)(3/2) = 1/3$. Now $\overline{UV}$ is parallel to the base $\overline{BP}$ of $\triangle RBP$ and it cuts side $\overline{BR}$ at point U, which is 1/3 of the way from B to R. It follows, therefore, that this line cuts side $\overline{PR}$ at a point $1/3$ of the way from P to R. In other words, $\overline{UV}$ goes through X.

Similarly, $CV/CQ = (CV/CA)(CA/CQ) = (2/9)(3) = 2/3$, and thus V lies $2/3$ of the way from C to Q along the side $\overline{CQ}$ of $\triangle QCP$. It follows that $\overline{UV}$ cuts side $\overline{PQ}$ at the point $2/3$ of the way from P to Q. In other words, $\overline{UV}$ goes through Y, and this completes the proof.

3. Let m and n be positive integers and assume that the number $A = \dfrac{(m+n)^3}{n^2}$ is an odd integer. Find the smallest possible value that A can have, and find all pairs m and n which yield this value. Justify your answer.

SOLUTION. Let d be the largest positive integer which divides both m and n, and write $m = da$, $n = db$. Then $A = (m+n)^3/n^2 = d^3(a+b)^3/d^2b^2 = d(a+b)^3/b^2$. Since b and a have no nontrivial common factor, the same is true of b and $a+b$, and hence of b^2 and $(a+b)^3$. Thus, in order for A to be an integer, b^2 must divide d. In particular, if $d = cb^2$, then $A = c(a+b)^3$ with a, b, c positive integers. Since $a+b \geq 2$ and A is odd, the minimum value of A is 27 which occurs when $c = 1$ and $a+b = 3$. Note that there are two possibilities for m and n in this case. First, if $a = 2$ and $b = 1$, then $d = cb^2 = 1$ so $m = da = 2$ and $n = db = 1$. On the other hand, if $a = 1$ and $b = 2$, then $d = cb^2 = 4$, so $m = da = 4$ and $n = db = 8$.

4. Given a square with side of length 1, find the maximum number of points that can be placed in (or on the boundary of) the square such that no two of the points are closer than $1/2$ unit apart. Explain why your solution is correct.

SOLUTION. We can easily place 9 points in the square so that no two of them are closer that $1/2$ unit. Put 4 points at the corners, 4 points at the midpoints of the sides and one point at the "center", where the diagonals of the square cross. It is easy to see that the minimum distance between any two of these 9 points is exactly $1/2$.

To show that it is not possible to place 10 points so that no two of them are closer than $1/2$ unit, divide the square into 9 small squares, each with sides of length $1/3$. (This can clearly be done by cutting each side of the original square into thirds and then joining the appropriate cut points.) Now, no matter how we place 10 points in the original square, there will necessarily be at least two of them in the same small square, and thus the distance between these two points will be at most the length of the diagonal of the small square, which is $\sqrt{2}/3$. Since $\sqrt{2}/3 < 1/2$, we see that it is not possible to place 10 points in a square of side 1 unless two of them are closer apart than $1/2$ unit. Thus, 9 points is the correct answer.

5. In Problem Set I, we saw that a power of 2 can never be written as a sum of two or more consecutive positive integers. Every power of 3, however, can be written in this way. Find, with appropriate justification, the largest integer m such that 3^{11} can be written as a sum of exactly m consecutive positive integers.

SOLUTION. Let $k+1, k+2, k+3, \ldots, k+m$ be a sequence of m consecutive positive integers, and note that the sum of these m integers is $mk + (1+2+\cdots+m) = mk + m(m+1)/2$. In particular, if this sequence sums to 3^{11}, then we have the equation $3^{11} = mk + m(m+1)/2$, or equivalently $2{\cdot}3^{11} = m(2k+m+1)$. It follows that m must be a factor of $2{\cdot}3^{11}$, and we see that the other factor, which is $2k+m+1$, must exceed m. If we try to factor $2{\cdot}3^{11} = 6{\cdot}3^{10}$ into integers in such a way that the smaller factor m is as large as possible, it is clear that we should take $m = 2{\cdot}3^5$, in which case the second factor would be $3{\cdot}3^5 = 3^6$. In order for this to actually solve our problem, there must exist some integer $k \geq 0$ such that $2k + 2{\cdot}3^5 + 1 = 2k + m + 1 = 3{\cdot}3^5$. But then, $k = (3^5 - 1)/2 = 121$ and therefore 3^{11} is the sum of the $m = 2{\cdot}3^5 = 486$ consecutive integers beginning with $k+1 = 122$.