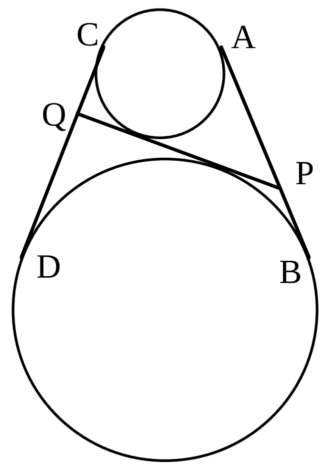
WISCONSIN MATHEMATICS SCIENCE & ENGINEERING TALENT SEARCH

PROBLEM SET III (1998-99) **DECEMBER 1998**

1. An operation $\square$ is defined on the set of positive integers, so that if x and y are any two positive integers, then $x \square y$ is also a positive integer. Assuming that $\square$ satisfies the three conditions listed below, for all positive integers x, y and z, compute $5 \square 9$ and justify your answer.

$$x \square (y+z) = (x \square y)(x \square z)$$
$$(x+y) \square 1 = (x \square 1) + (y \square 1)$$
$$(x+y) \square 2 = (x \square 2) + 4(xy \square 1) + (y \square 2)\,.$$



2. In the figure, the line segments $\overline{AB}$, $\overline{CD}$ and $\overline{PQ}$ are common tangents to two given circles, where points A and C are on one of the circles, B and D are on the other circle and points P and Q are on $\overline{AB}$ and $\overline{CD}$, as shown. Prove that $PB = QC$

3. (New Year's Problem) If $n > 1$ is an integer and if we write

$$S_n = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}}\,,$$

show that $2\sqrt{n+1} - 2 < S_n < 2\sqrt{n} - 1$. Deduce that the number $S_{1,000,000}$ lies between 1998 and 1999.

4. Let x, y and z be positive real numbers. Show that $(x+y)(x+z)(y+z) \geq 8xyz$.

5. We construct a sequence of numbers $A_1, A_2, A_3, \ldots$ in such a way that $A_n + A_{n+1} = A_{n+2}$ for all subscripts $n \geq 1$. Suppose that $A_2 = 3$ and $A_{50} = 300$. Compute the value of the sum $S = A_1 + A_2 + A_3 + \cdots + A_{48}$, and justify your answer.

You are invited to submit a solution even if you get just one problem. Please do not write your solutions on the page. Remember that solutions usually require a proof or justification.

RETURN TO: MATHEMATICS TALENT SEARCH
Dept. of Mathematics, 480 Lincoln Drive
University of Wisconsin, Madison, WI 53706

DEADLINE
January 8
1999

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(Please Detach)

Last Name First Name Grade

School Town

Home Address Town Zip Code

PROBLEM	SCORE
1	
2	
3	
4	
5	

PROBLEM SET III