1. An operation $\square$ is defined on the set of positive integers, so that if $x$ and $y$ are any two positive integers, then $x \square y$ is also a positive integer. Assuming that $\square$ satisfies the three conditions listed below, for all positive integers $x$, $y$ and $z$, compute $5 \square 9$ and justify your answer.

\[
x \square (y + z) = (x \square y)(x \square z)\]
\[
(x + y) \square 1 = (x \square 1) + (y \square 1)\]
\[
(x + y) \square 2 = (x \square 2) + 4(xy \square 1) + (y \square 2).
\]

2. In the figure, the line segments $AB$, $CD$ and $PQ$ are common tangents to two given circles, where points $A$ and $C$ are on one of the circles, $B$ and $D$ are on the other circle and points $P$ and $Q$ are on $AB$ and $CD$, as shown. Prove that $PB = QC$.

3. (New Year’s Problem) If $n > 1$ is an integer and if we write

\[
S_n = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}},
\]

show that $2\sqrt{n+1} - 2 < S_n < 2\sqrt{n} - 1$. Deduce that the number $S_{1,000,000}$ lies between 1998 and 1999.

4. Let $x$, $y$ and $z$ be positive real numbers. Show that $(x + y)(x + z)(y + z) \geq 8xyz$.

5. We construct a sequence of numbers $A_1, A_2, A_3, \ldots$ in such a way that $A_n + A_{n+1} = A_{n+2}$ for all subscripts $n \geq 1$. Suppose that $A_2 = 3$ and $A_{50} = 300$. Compute the value of the sum $S = A_1 + A_2 + A_3 + \cdots + A_{48}$, and justify your answer.

You are invited to submit a solution even if you get just one problem. Please do not write your solutions on the problem set page. Remember that solutions usually require a proof or justification.