WISCONSIN MATHEMATICS SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET III (1998-99)

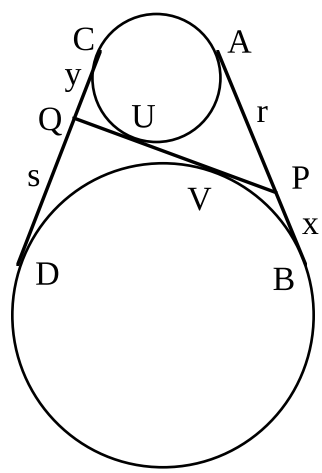
1. An operation $\Box$ is defined on the set of positive integers, so that if x and y are any two positive integers, then $x \Box y$ is also a positive integer. Assuming that $\Box$ satisfies the three conditions listed below, for all positive integers x, y and z, compute $5 \Box 9$ and justify your answer.

$$x \Box (y+z) = (x \Box y)(x \Box z)$$
$$(x+y) \Box 1 = (x \Box 1) + (y \Box 1)$$
$$(x+y) \Box 2 = (x \Box 2) + 4(xy \Box 1) + (y \Box 2)\,.$$

SOLUTION. By the second equation above, we get $2 \Box 1 = (1 \Box 1) + (1 \Box 1) = 2a$ where $a = 1 \Box 1$. Similarly, $3 \Box 1 = (1 \Box 1) + (2 \Box 1) = a + 2a = 3a$, and continuing like this, we see that $n \Box 1 = na$ for every positive integer n. We can now use the first of the given equations to show that $n \Box 2 = (n \Box 1)(n \Box 1) = (na)^2$. Similarly, $n \Box 3 = (n \Box 1)(n \Box 2) = (na)(na)^2 = (na)^3$, and continuing like this, we deduce that $n \Box m = (na)^m$ for all positive integers n and m. In particular, $2 \Box 2 = 4a^2$.

We can also compute $2 \Box 2$ from the third of the given equations. Indeed, if we plug in $x = 1 = y$, we get $2 \Box 2 = (1 \Box 2) + 4(1 \Box 1) + (1 \Box 2) = a^2 + 4a + a^2$. We now have $4a^2 = 2 \Box 2 = 2a^2 + 4a$, so $2a^2 = 4a$. Since $a \neq 0$, this yields $a = 2$, and thus $m \Box n = (2n)^m$. In particular, $5 \Box 9 = 10^9$, or 1 billion.

2. In the figure, the line segments $\overline{AB}$, $\overline{CD}$ and $\overline{PQ}$ are common tangents to two given circles, where points A and C are on one of the circles, B and D are on the other circle and points P and Q are on $\overline{AB}$ and $\overline{CD}$, as shown. Prove that $PB = QC$



SOLUTION. Let U and V denote the points of tangency of $\overline{PQ}$ with the two circles, as shown. Also, let $PB = x$, $QC = y$, $AP = r$ and $DQ = s$, as indicated in the figure. Note that $AB = CD$, so that we have $r + x = s + y$. Also, $PV = PB = x$ and $PU = PA = r$, and it follows that $UV = r - x$. Similarly, $QU = QC = y$ and $QV = QD = s$, and this yields $UV = s - y$. We therefore have $r - x = s - y$. If we subtract this from our earlier equation, which was $r + x = s + y$, we get $2x = 2y$, and thus $x = y$, as desired.

3. (New Year's Problem) If $n > 1$ is an integer and if we write

$$S_n = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}}\,,$$

show that $2\sqrt{n+1} - 2 < S_n < 2\sqrt{n} - 1$. Deduce that the number $S_{1,000,000}$ lies between 1998 and 1999.

SOLUTION. First, we show that $2\sqrt{n+1} - 2 < S_n$ for all integers $n \geq 1$. This inequality certainly holds when $n = 1$ since $2\sqrt{2} - 2 < 1$. We will show that if the desired inequality holds for some value of n, then it automatically holds for the next value of n. Since we know that it holds for $n = 1$, it will follow that it holds for $n = 2$, and thus for $n = 3$, and so on, for all positive integers n. (In other words, we are using the principle of mathematical induction.)

We assume, therefore, that $2\sqrt{n+1} - 2 < S_n$, and we add $1/\sqrt{n+1}$ to both sides to get

$$2\sqrt{n+1} - 2 + \frac{1}{\sqrt{n+1}} < S_n + \frac{1}{\sqrt{n+1}} = S_{n+1}\,.$$

Observe that $(2\sqrt{n+1} + 1/\sqrt{n+1})^2 > 4(n+1) + 4 = (2\sqrt{n+2})^2$. It therefore follows that $2\sqrt{n+1} + 1/\sqrt{n+1} > 2\sqrt{n+2}$, and consequently

$$S_{n+1} > 2\sqrt{n+1} + \frac{1}{\sqrt{n+1}} - 2 > 2\sqrt{n+2} - 2\,.$$

Similarly, we prove by induction that $S_n < 2\sqrt{n} - 1$ for every integer $n > 1$. This certainly holds when $n = 2$, and we assume that it holds for some particular value of n, so that $S_n < 2\sqrt{n} - 1$. Adding $1/\sqrt{n+1}$ to both sides yields

$$S_{n+1} < 2\sqrt{n} + \frac{1}{\sqrt{n+1}} - 1\,.$$

But $(2\sqrt{n+1} - 1/\sqrt{n+1})^2 > 4(n+1) - 4 = (2\sqrt{n})^2$, so $2\sqrt{n+1} - 1/\sqrt{n+1} > 2\sqrt{n}$, and thus

$$S_{n+1} < 2\sqrt{n} + \frac{1}{\sqrt{n+1}} - 1 < 2\sqrt{n+1} - 1\,,$$

as desired. Finally, we plug in $n = 1,000,000$ and obtain $1998 < S_{1,000,000} < 1999$.

4. Let x, y and z be positive real numbers. Show that $(x+y)(x+z)(y+z) \geq 8xyz$.

SOLUTION. An easy computation shows that

$$(x+y)(x+z)(y+z) = 2xyz + x(y^2+z^2) + y(x^2+z^2) + z(x^2+y^2)\,.$$

Now $y^2 + z^2 - 2yz = (y-z)^2 \geq 0$, and thus $y^2 + z^2 \geq 2yz$. Since $x > 0$, this yields $x(y^2+z^2) \geq 2xyz$. Similarly, we get $y(x^2+z^2) \geq 2xyz$ and $z(x^2+y^2) \geq 2xyz$. Combining these inequalities, we get $(x+y)(x+z)(y+z) \geq 8xyz$, as desired.

5. We construct a sequence of numbers $A_1, A_2, A_3, \ldots$ in such a way that $A_n + A_{n+1} = A_{n+2}$ for all subscripts $n \geq 1$. Suppose that $A_2 = 3$ and $A_{50} = 300$. Compute the value of the sum $S = A_1 + A_2 + A_3 + \cdots + A_{48}$, and justify your answer.

SOLUTION. For all integers $m \leq 48$, write $S_m = A_m + A_{m+1} + A_{m+2} + \cdots + A_{48}$, so that the quantity S to be evaluated is actually S_1. Observe that $A_{m+1} + S_m = A_{m+1} + A_m + S_{m+1} = A_{m+2} + S_{m+1}$. In particular, taking $m = 1$, we have $A_2 + S_1 = A_3 + S_2$. If we take $m = 2$, we get $A_3 + S_2 = A_4 + S_3$. Continuing like this, we get

$$A_2 + S_1 = A_3 + S_2 = A_4 + S_3 = \cdots = A_{49} + S_{48}\,.$$

But $S_{48} = A_{48}$, so this yields $A_2 + S_1 = A_{49} + A_{48} = A_{50} = 300$. Since $A_2 = 3$, we conclude that $S = S_1 = 297$.