WISCONSIN MATHEMATICS SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET III (1998-99)

1. An operation $\square$ is defined on the set of positive integers, so that if $x$ and $y$ are any two positive integers, then $x \square y$ is also a positive integer. Assuming that $\square$ satisfies the three conditions listed below, for all positive integers $x$, $y$ and $z$, compute $5 \square 9$ and justify your answer.

   \[
   x \square (y + z) = (x \square y)(x \square z) \\
   (x + y) \square 1 = (x \square 1) + (y \square 1) \\
   (x + y) \square 2 = (x \square 2) + 4(xy \square 1) + (y \square 2).
   \]

   **SOLUTION.** By the second equation above, we get $2 \square 1 = (1 \square 1) + (1 \square 1) = 2a$ where $a = 1 \square 1$. Similarly, $3 \square 1 = (1 \square 1) + (2 \square 1) = a + 2a = 3a$, and continuing like this, we see that $n \square 1 = na$ for every positive integer $n$. We can now use the first of the given equations to show that $n \square 2 = (n \square 1)(n \square 1) = (na)^2$. Similarly, $n \square 3 = (n \square 1)(n \square 2) = (na)(na)^2 = (na)^3$, and continuing like this, we deduce that $n \square m = (na)^m$ for all positive integers $n$ and $m$. In particular, $2 \square 2 = 4a^2$.

   We can also compute $2 \square 2$ from the third of the given equations. Indeed, if we plug in $x = 1 = y$, we get $2 \square 2 = (1 \square 2) + 4(1 \square 1) + (1 \square 2) = a^2 + 4a + a^2$. We now have $4a^2 = 2 \square 2 = 2a^2 + 4a$, so $2a^2 = 4a$. Since $a \neq 0$, this yields $a = 2$, and thus $m \square n = (2n)^m$. In particular, $5 \square 9 = 10^9$, or 1 billion.

2. In the figure, the line segments $AB$, $CD$ and $PQ$ are common tangents to two given circles, where points $A$ and $C$ are on one of the circles, $B$ and $D$ are on the other circle and points $P$ and $Q$ are on $AB$ and $CD$, as shown. Prove that $PB = QC$.

   **SOLUTION.** Let $U$ and $V$ denote the points of tangency of $PQ$ with the two circles, as shown. Also, let $PB = x$, $QC = y$, $AP = r$ and $DQ = s$, as indicated in the figure. Note that $AB = CD$, so that we have $r + x = s + y$. Also, $PV = PB = x$ and $PU = PA = r$, and it follows that $UV = r - x$. Similarly, $QU = QC = y$ and $QV = QD = s$, and this yields $UV = s - y$. We therefore have $r - x = s - y$. If we subtract this from our earlier equation, which was $r + x = s + y$, we get $2x = 2y$, and thus $x = y$, as desired.

3. (New Year’s Problem) If $n > 1$ is an integer and if we write

   \[
   S_n = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}},
   \]

   show that $2\sqrt{n} + 1 - 2 < S_n < 2\sqrt{n} - 1$. Deduce that the number $S_{1,000,000}$ lies between 1998 and 1999.

   **SOLUTION.** First, we show that $2\sqrt{n} + 1 - 2 < S_n$ for all integers $n \geq 1$. This inequality certainly holds when $n = 1$ since $2\sqrt{2} - 2 < 1$. We will show that if the desired inequality holds for some value of $n$, then it automatically holds for the next value of $n$. Since we know that it holds
for \( n = 1 \), it will follow that it holds for \( n = 2 \), and thus for \( n = 3 \), and so on, for all positive integers \( n \). (In other words, we are using the principle of mathematical induction.)

We assume, therefore, that \( 2\sqrt{n+1} - 2 < S_n \), and we add \( 1/\sqrt{n+1} \) to both sides to get

\[
2\sqrt{n+1} - 2 + \frac{1}{\sqrt{n+1}} < S_n + \frac{1}{\sqrt{n+1}} = S_{n+1}.
\]

Observe that \((2\sqrt{n+1} + 1/\sqrt{n+1})^2 > 4(n+1) + 4 = (2\sqrt{n} + 2)^2\). It therefore follows that \(2\sqrt{n+1} + 1/\sqrt{n+1} > 2\sqrt{n} + 2\), and consequently

\[
S_{n+1} > 2\sqrt{n+1} + \frac{1}{\sqrt{n+1}} - 2 > 2\sqrt{n+2} - 2.
\]

Similarly, we prove by induction that \( S_n < 2\sqrt{n} - 1 \) for every integer \( n > 1 \). This certainly holds when \( n = 2 \), and we assume that it holds for some particular value of \( n \), so that \( S_n < 2\sqrt{n} - 1 \). Adding \( 1/\sqrt{n+1} \) to both sides yields

\[
S_{n+1} < 2\sqrt{n} + \frac{1}{\sqrt{n+1}} - 1.
\]

But \((2\sqrt{n+1} - 1/\sqrt{n+1})^2 > 4(n+1) - 4 = (2\sqrt{n})^2\), so that \(2\sqrt{n+1} - 1/\sqrt{n+1} > 2\sqrt{n}\), and thus

\[
S_{n+1} < 2\sqrt{n} + \frac{1}{\sqrt{n+1}} - 1 < 2\sqrt{n+1} - 1,
\]

as desired. Finally, we plug in \( n = 1, 000, 000 \) and obtain \( 1998 < S_{1,000,000} < 1999 \).

4. Let \( x, y \) and \( z \) be positive real numbers. Show that \((x+y)(x+z)(y+z) \geq 8xyz\).

**SOLUTION.** An easy computation shows that

\[
(x+y)(x+z)(y+z) = 2xyz + x(y^2 + z^2) + y(x^2 + z^2) + z(x^2 + y^2).
\]

Now \(y^2 + z^2 - 2yz = (y-z)^2 \geq 0\), and thus \(y^2 + z^2 \geq 2yz\). Since \(x > 0\), this yields \(x(y^2 + z^2) \geq 2xyz\). Similarly, we get \(y(x^2 + z^2) \geq 2xyz\) and \(z(x^2 + y^2) \geq 2xyz\). Combining these inequalities, we get \((x+y)(x+z)(y+z) \geq 8xyz\), as desired.

5. We construct a sequence of numbers \( A_1, A_2, A_3, \ldots \) in such a way that \( A_n + A_{n+1} = A_{n+2} \) for all subscripts \( n \geq 1 \). Suppose that \( A_2 = 3 \) and \( A_{50} = 300 \). Compute the value of the sum \( S = A_1 + A_2 + A_3 + \cdots + A_{48} \), and justify your answer.

**SOLUTION.** For all integers \( m \leq 48 \), write \( S_m = A_m + A_{m+1} + A_{m+2} + \cdots + A_{48} \), so that the quantity \( S_m \) to be evaluated is actually \( S_1 \). Observe that \( A_{m+1} + S_m = A_{m+1} + A_m + S_{m+1} = A_{m+2} + S_{m+1} \). In particular, taking \( m = 1 \), we have \( A_2 + S_1 = A_3 + S_2 \). If we take \( m = 2 \), we get \( A_3 + S_2 = A_4 + S_3 \). Continuing like this, we get

\[
A_2 + S_1 = A_3 + S_2 = A_4 + S_3 = \cdots = A_{49} + S_{48}.
\]

But \( S_{48} = A_{48} \), so this yields \( A_2 + S_1 = A_{49} + A_{48} = A_{50} = 300 \). Since \( A_2 = 3 \), we conclude that \( S = S_1 = 297 \).