WISCONSIN MATHEMATICS SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET II (1998-99)

1. Find all prime numbers p for which it is possible to write

$$\frac{1}{p} = \frac{1}{a^2} + \frac{1}{b^2}$$

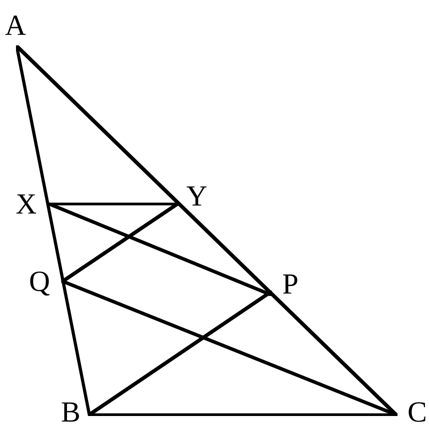
with positive integers a and b.

SOLUTION. We can have $p = 2$ by setting $a = 2 = b$, and we will show that the equation

$$\frac{1}{p} = \frac{1}{a^2} + \frac{1}{b^2}$$

cannot hold if $p > 2$. Clearing denominators from the equation, we obtain $a^2b^2 = pb^2 + pa^2 = p(a^2 + b^2)$. It follows that a^2b^2 must be a multiple of the prime p, and thus either a or b is a multiple of p. Also, since a^2b^2 is a square integer, it must, in fact, be divisible by p^2. Hence p^2 divides $p(a^2 + b^2)$. It follows that $a^2 + b^2$ is a multiple of p. We know however, that at least one of a^2 or b^2 is a multiple of p, so it follows that the other is too. We now know that each of a and b is a multiple of p, and thus $1/a^2 \le 1/p^2$, and similarly, $1/b^2 \le 1/p^2$. It now follows from the original equation that $1/p \le 2/p^2$, and from this we deduce that $p \le 2$.

2. Points P and Q are selected on sides $\overline{AC}$ and $\overline{AB}$ of $\triangle ABC$, and line segments $\overline{BP}$ and $\overline{CQ}$ are drawn. Then $\overline{PX}$ and $\overline{QY}$ are drawn parallel to $\overline{CQ}$ and $\overline{BP}$, as shown. Prove that $\overline{XY}$ is parallel to $\overline{BC}$.



SOLUTION. In $\triangle ABP$, we see that $\overline{QY}$ is parallel to the base $\overline{BP}$, and thus $\triangle AQY \sim \triangle ABP$, so we deduce that $AY/AP = AQ/AB$. Similarly, $\triangle APX \sim \triangle ACQ$, and we have $AX/AQ = AP/AC$. These two equations yield

$$AY{\cdot}AB = AP{\cdot}AQ = AX{\cdot}AC\,,$$

and thus $AY/AC = AX/AB$. It follows that $\triangle AXY \sim \triangle ABC$ by the SAS similarity criterion, and thus $\angle AXY = \angle ABC$. We now see that $\overline{XY} \| \overline{BC}$, as required.

3. Your calculator will tell you that the quantity

$$z = \left(2{\cdot}\sqrt[3]{2} + 1 - \sqrt{12{\cdot}\sqrt[3]{2} - 15}\right)^3$$

is approximately an integer. Is z *exactly* an integer? Justify your answer.

SOLUTION. According to the calculator, the number z is approximately equal to 32, and so we would like to decide whether or not the quantity

$$y = 2{\cdot}\sqrt[3]{2} + 1 - \sqrt{12{\cdot}\sqrt[3]{2} - 15}$$

is exactly equal to $\sqrt[3]{32}$. For notational simplicity, we will write $\alpha = \sqrt[3]{2}$ here, so that $y = 2\alpha + 1 - \sqrt{12\alpha - 15}$. Also, since $32 = 2^5$, we see that $\sqrt[3]{32} = 2\alpha^2$, and hence we need to determine whether or not the equation

$$\sqrt{12\alpha - 15} = 2\alpha + 1 - 2\alpha^2$$

is valid. Using a calculator, we see that the right side of this equation exceeds 0.3, and so it is definitely positive. We can thus check the equation by showing that the squares of both sides are equal. If we square the right side and combine like terms, we get $4\alpha^4 - 8\alpha^3 + 4\alpha + 1$. But since $\alpha^3 = 2$, we see that $\alpha^4 = 2\alpha$, and thus $4\alpha^4 - 8\alpha^3 + 4\alpha + 1 = 12\alpha - 15$. Consequently, our equation is true, and z is indeed exactly equal to 32.

4. If x, y and z are positive numbers, show that

$$8(x^3 + y^3 + z^3) \geq (x + y)^3 + (x + z)^3 + (y + z)^3 .$$

SOLUTION. We begin by computing

$$\begin{aligned} 4(x^3 + y^3) - (x + y)^3 &= (x + y)\big(4(x^2 - xy + y^2) - (x + y)^2\big) \\ &= (x + y)(3x^2 - 6xy + 3y^2) \\ &= 3(x + y)(x - y)^2 \geq 0 , \end{aligned}$$

where the inequality holds since x and y are assumed to be positive, so $x + y > 0$ and of course $(x - y)^2 \geq 0$. Thus $4x^3 + 4y^3 \geq (x + y)^3$ and similar calculations show that $4x^3 + 4z^3 \geq (x + z)^3$ and $4y^3 + 4z^3 \geq (y + z)^3$. Adding these three inequalities, we obtain

$$8x^3 + 8y^3 + 8z^3 \geq (x + y)^3 + (x + z)^3 + (y + z)^3 ,$$

which is precisely what we wanted.

5. Let S be a subset of the set $\{1, 2, 3, \ldots, 99, 100\}$, and assume that S has at least ten members. Show that there are nonempty subsets X and Y of S such that X and Y have no members in common and the sum of the members of X is equal to the sum of the members of Y.

SOLUTION. First, we observe that the total number of subsets of S, counting the empty subset is 2^n, where n is the size of S. To see why this is so, notice that each subset X of S is completely determined by the answers to the n yes-or-no questions: "Is s a member of X?" for each s in S.

Next, we remark that it is no loss to assume that S has exactly ten members since we can simply delete any excess elements. Thus S has exactly $2^{10} - 1 = 1023$ nonempty subsets. The largest possible sum for any of these subsets cannot exceed the sum of the ten largest numbers in the set $\{1, 2, 3, \ldots, 99, 100\}$, and this maximum sum is clearly less than 1000. Consequently, there are fewer than 1000 different sums that can occur. Since there are more than 1000 subsets and fewer than 1000 possible sums, the *Pigeonhole Principle* implies that there must exist two different subsets X and Y with equal sums. This does not quite solve the problem, however, since we were asked to find two subsets with no members in common for which the sums are equal. We can accomplish this simply by deleting from X and Y all common members. This diminishes the sums for each of X and Y by the same amount, and thus the two new sums are still equal. Also, after deleting the overlap, both sets remain nonempty since neither of the original sets X and Y can be contained in the other because their sums are equal.