WISCONSIN MATHEMATICS SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET I (1998-99)

1. If x and y are any two real numbers, show that

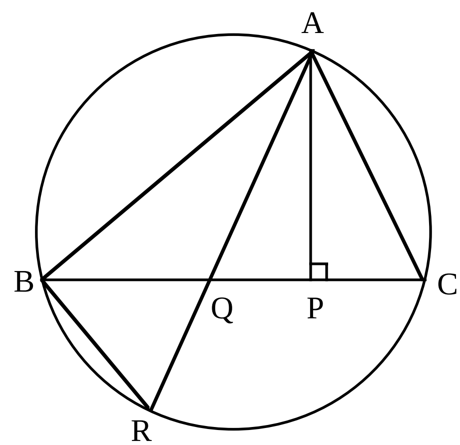
$$(x^2+y^2)^2 \geq xy(x+y)^2\,.$$

SOLUTION. We prove the inequality by showing that the left side minus the right side is always greater than or equal to zero. We compute

$$\begin{aligned}(x^2+y^2)^2 - xy(x+y)^2 &= x^4+2x^2y^2+y^4-x^3y-2x^2y^2-xy^3\\ &= x^4-x^3y+y^4-xy^3\\ &= x^3(x-y)+y^3(y-x)\\ &= (x-y)(x^3-y^3)\,.\end{aligned}$$

Finally, it is easy to see that $(x-y)(x^3-y^3) \geq 0$ since either both factors are positive or both factors are negative or both factors are zero.

2. In the diagram, $\overline{AP}$ is an altitude of $\triangle ABC$, and point Q is chosen on side $\overline{BC}$ so that $\angle BAQ = \angle CAP$. Show that line $\overline{AQ}$ goes through the circumcenter of $\triangle ABC$.



SOLUTION. Draw the circumcircle and extend $\overline{AQ}$ to meet the circle at R. Then draw chord $\overline{BR}$, as shown. We must show that $\overline{AR}$ is a diameter of the circle, and for this it suffices to show that $\angle ABR = 90^\circ$. Comparing $\triangle APC$ and $\triangle ABR$, we see that $\angle BAR = \angle PAC$ by hypothesis, and $\angle ACP = \angle ARB$ because these angles subtend the same arc on the circle. It follows that the third angles of the two triangles are equal, and we have $\angle ABR = \angle APC = 90^\circ$, as desired.

3. Observe that

$$\begin{aligned}2{\cdot}1^2 &= 1^2+1\\ 2{\cdot}2^2 &= 3^2-1\\ 2{\cdot}5^2 &= 7^2+1\\ 2{\cdot}12^2 &= 17^2-1\,.\end{aligned}$$

Show that there are actually infinitely many positive integer solutions to each of the equations $2x^2 = y^2+1$ and $2x^2 = y^2-1$.

SOLUTION. Note the pattern here. It appears that if we have $2a^2 = b^2+e$ on one line, where either $e=1$ or $e=-1$, then the left side of the next line is $2(a+b)^2$ and the right side of the next line is $(2a+b)^2-e$. If this pattern continues, we would expect that the line following the last printed equation would read $2(29)^2 = 41^2+1$, and sure enough, this is true.

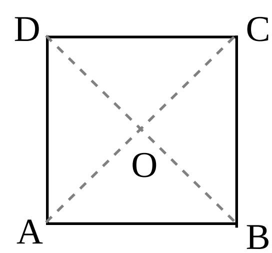
If we can show that $2(a+b)^2 = (2a+b)^2 - e$ whenever $2a^2 = b^2 + e$, then this will show that the pattern continues indefinitely, and there really are infinitely many solutions to each of the two equations $2x^2 = y^2 + 1$ and $2x^2 = y^2 - 1$.

Thus suppose that $2a^2 = b^2 + e$. Then

$$\begin{aligned} 2(a+b)^2 - ((2a+b)^2 - e) &= 2a^2 + 4ab + 2b^2 - (4a^2 + 4ab + b^2 - e) \\ &= b^2 - 2a^2 + e = 0 \end{aligned}.$$

and $2(a+b)^2 = (2a+b)^2 - e$, as we wanted.

4. Suppose that every point in the plane is colored either red or blue. Prove that there exists an isosceles right triangle all of whose vertices have the same color.



SOLUTION. Let A and B be two points of the same color, say red. Draw the square $ABCD$ as indicated and let O be its center. If point C is red, then $\triangle ABC$ is an isosceles right triangle with all vertices red, and we are done. We can thus assume that C is colored blue, and similarly, we can assume that D is blue. Finally, if O is red, then $\triangle AOB$ is an isosceles right triangle with all vertices red, and if O is blue, then $\triangle COD$ is an isosceles right triangle with all vertices blue. This completes the proof.

5. Prove that no power of 2 can be written as a sum of two or more consecutive positive integers.

SOLUTION. The sum of the n consecutive integers starting with m is

$$m + (m+1) + \cdots + (m+n-1) = nm + n(n-1)/2$$

and if this is equal to 2^e, we have $n(2m+n-1) = 2^{e+1}$. If this equation holds with $n > 1$, we see that n must be even because it is a divisor of 2^{e+1}. But then $2m+n-1$ is an odd positive divisor of 2^{e+1}, and it follows that $2m+n-1 = 1$. This yields $m = (2-n)/2$, and since $n \geq 2$, we see that $m \leq 0$ and thus not all of our consecutive integers are positive. This is the required contradiction.