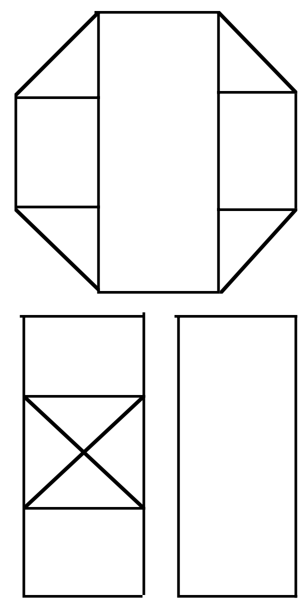
WISCONSIN MATHEMATICS SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET V (1997-98)

1. Let r be a real number with $-6 < r < 6$. Find all real numbers a, b, c, d that satisfy $x^4 + rx^2 + 9 = (x^2 + ax + b)(x^2 + cx + d)$, as polynomials in the variable x.

SOLUTION. First, by matching the coefficients of x^3 on both sides of the equation, we get $0 = a + c$, so $c = -a$. Then, by matching the coefficients of x, we get $0 = ad + bc = a(d - b)$. If $a = 0$, then $x^4 + rx^2 + 9 = (x^2 + b)(x^2 + d)$, so $b + d = r$ and $bd = 9$. Hence $(b - d)^2 = (b + d)^2 - 4bd = r^2 - 36 < 0$ since $-6 < r < 6$, and this contradicts the fact that b and d are real. Therefore $a \neq 0$ and the equation $0 = a(b - d)$ implies that $b = d$. We now have $x^4 + rx^2 + 9 = (x^2 + ax + b)(x^2 - ax + b) = x^4 + (2b - a^2)x^2 + b^2$, so $b^2 = 9$, $b = \pm 3$, and $2b - a^2 = r$. In particular, this yields $-6 < r < 2b$, so $b \neq -3$. It follows that $b = 3$ and $a^2 = 6 - r$. In other words, $d = b = 3$ and $c = -a = \pm\sqrt{6 - r}$.

2. Cut a regular octagon into seven pieces so that these pieces can be put together in one way to form a rectangle and then in another way to form a second rectangle of a different shape.



SOLUTION. Cut the octagon as indicated by the solid lines, and note that the four triangular pieces can be combined to form a square which fits precisely in the center of the octagon. By combining this square with the two side panels, we obtain a rectangular strip which is congruent to large seventh piece. These two strips can, of course, be joined either end-to-end or side-to-side to form rectangles of different shapes.

3. Let q be a positive rational number. Show that there are only finitely many positive integers n_1, n_2, n_3, n_4 with

$$\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} + \frac{1}{n_4} = q.$$

(Of course, there may be no solutions.)

SOLUTION. First note that the equation $1/n_1 = q$ (for any rational number q) has at most one solution. Next, consider $1/n_1 + 1/n_2 = q$. If $n_2 \leq n_1$, then $2/n_2 \geq 1/n_1 + 1/n_2 = q$, and hence $n_2 \leq 2/q$. Thus, there are only finitely many possibilities for n_2 and with each such choice, there is at most one n_1 satisfying $1/n_1 = q - 1/n_2$. In other words, the equation $1/n_1 + 1/n_2 = q$ has only finitely many solutions. Now let's move up to $1/n_1 + 1/n_2 + 1/n_3 = q$ and suppose that $n_3 \leq n_2 \leq n_1$. Then $3/n_3 \geq 1/n_1 + 1/n_2 + 1/n_3 = q$, so $n_3 \leq 3/q$ and there are only finitely many possibilities for n_3. Moreover, with each choice of n_3, we know that the equation $1/n_1 + 1/n_2 = q - 1/n_3$ has only finitely many solutions, and therefore there are only finitely many solutions all told for $1/n_1 + 1/n_2 + 1/n_3 = q$. Finally, we study the given equation $1/n_1 + 1/n_2 + 1/n_3 + 1/n_4 = q$ and assume, for convenience, that $n_4 \leq n_3 \leq n_2 \leq n_1$. Then $4/n_4 \geq 1/n_1 + 1/n_2 + 1/n_3 + 1/n_4 = q$, so $n_4 \leq 4/q$ and there are only finitely many possibilities for n_4. Furthermore, with each such choice, we now know that there are only finitely many possibilities for n_1, n_2 and n_3 satisfying $1/n_1 + 1/n_2 + 1/n_3 = q - 1/n_4$. We conclude, therefore, that only finitely many solutions exist.

4. We wish to fill the 16 boxes of the 4×4 square array with the letters a, b, c and d in such a way that each letter appears precisely once in each row and precisely once in each column. In how many different ways can this be done?

SOLUTION. A configuration as described above is called a Latin Square, and it is clear that we can permute the rows of a Latin Square or the columns of a Latin Square and still preserve the property that each letter appears precisely once in each row and precisely once in each column. Suppose that the first row and first column are as indicated. Then we can use these entries as coordinates to label the remaining positions. For example, we will use (c, d) to denote the entry in the row starting with c and in the column starting with d.

a	b	c	d
b			
c			
d			

Now, $(b, b) \neq b$. If $(b, b) = a$, then $(b, c) = d$, $(b, d) = c$, $(c, b) = d$ and $(d, b) = c$. With this, it follows easily that precisely two configurations can occur. On the other hand, if $(b, b) = c$, then $(b, d) = a$, $(b, c) = d$, $(d, b) = a$, $(c, b) = d$, and this yields just one configuration. Similarly, $(b, b) = d$ yields one possibility, so there are four Latin Squares all told with the first row and column as indicated. Next, just suppose that the first row is as indicated. Then there are $3! = 6$ ways to permute the second, third and fourth rows of the array, and all of these give rise to a different first column, and to four different Latin Squares. In other words, there are $6 \cdot 4 = 24$ Latin Squares with first row $\fbox{a|b|c|d}$. Finally, an arbitrary first row must contain the letters a, b, c, d in some order. Furthermore, there are $4! = 24$ permutations of the four columns, and each of these gives rise to a different first row and to 24 different Latin Squares. Thus, all told, there are $24 \cdot 24 = 576$ such Latin Squares.

5. For any positive integer n, let $f(n) = n + 1$ if n is odd and $f(n) = n/2$ if n is even. Now let $g(n)$ be the smallest number of times that the function f must be applied repeatedly to n until 1 is reached. For instance, $g(1) = 0$, $g(2) = 1$, $g(3) = 3$ (since $f(f(f(3))) = 1$), and $g(4) = 2$ (since $f(f(4)) = 1$). Find a general rule for determining $g(n)$ from n.

SOLUTION. If n is a power of 2, say $n = 2^m$, then clearly $g(n) = m$. Thus, it suffices to assume in the remainder of this answer that n is not a power of 2 and, for such integers, we offer the following formula for $g(n)$. To start with, let $d(n)$ be the number of digits in the binary representation of n and let $z(n)$ be the number of zeros in this representation that have a 1 digit somewhere to their right. We will show that $g(n) = d(n) + z(n) + 1$. It is easy to check this for small values of n. Now suppose that the formula holds for all appropriate numbers smaller than n. We will show that it holds for n as well. If n is even, say $n = 2k$, then $d(n) = d(k) + 1$ and $z(n) = z(k)$, since the binary representation of n is just that of k with a zero appended at the right. Since $f(n) = k$ and k is not a power of 2, it follows that $g(n) = g(k)+1 = d(k)+z(k)+1+1 = d(n)+z(n)+1$, proving the formula in this case. Next suppose that $n \neq 1$ is one less than a power of 2, say $n = 2^m - 1$. Then the binary representation of n is a string of m ones, so $d(n) = m$, and $z(n) = 0$. Also, f takes n to 2^m in one step and thence to 1 in m more steps, so $g(n) = m + 1 = d(n) + z(n) + 1$. Finally, suppose that $n = 2k - 1$ is odd, but not one less than a power of 2. Then the binary representation of n starts at the right with a sequence of 1's, followed by a 0. When we add one to n, all these 1's change to 0's, and the 0 becomes a 1. Thus $d(n) = d(n + 1) = d(k) + 1$ and $z(n) = z(n + 1) + 1 = z(k) + 1$. Since $f(f(n)) = k$, $k < n$ and k is not a power of 2, we have $g(n) = g(k) + 2 = d(k) + z(k) + 1 + 2 = d(n) + z(n) + 1$ and the formula is proved in all cases.