1. **Suppose that** \( a \) **and** \( b \) **are integers such that** \( a + 2b \) **and** \( b + 2a \) **are squares. Prove that each of** \( a \) **and** \( b \) **is a multiple of 3.**

**SOLUTION.** Write \( a + 2b = m^2 \) and \( 2a + b = n^2 \), where \( m \) **and** \( n \) **are integers. Then** \( m^2 + n^2 = 3a + 3b \), **and so** \( m^2 + n^2 \) **is a multiple of 3. We can write** \( m \) **in one of the three forms** \( m = 3k \), \( m = 3k + 1 \) **or** \( m = 3k - 1 \), **where** \( k \) **is some integer. In the first case,** \( m^2 \) **is a multiple of 9, and in the second and third cases,** we compute that \( m^2 = 9k^2 \pm 6k + 1 \), **and thus** \( m^2 \) **has the form** \( 3t + 1 \) **for some integer** \( t \). **Similarly,** **of** course,** \( n^2 \) **is either a multiple of 9 or else it has the form** \( 3s + 1 \) **for some integer** \( s \). **If either of** \( m^2 \) **or** \( n^2 \) **is not a multiple of 9,** it therefore follows that \( m^2 + n^2 \) **is either one or two more than a multiple of 3. However,** **we know that this is not the case,** **and consequently each of** \( m^2 \) **and** \( n^2 \) **is actually a multiple of 9.**

Since \( 3(a + b) = m^2 + n^2 \) **is a multiple of 9,** we deduce that \( a + b \) **must be a multiple of 3. Also,** \( a - b = n^2 - m^2 \) **is a multiple of 3. (In fact,** it **is a multiple of 9.)** It follows that \( 2a = (a + b) + (a - b) \) **is a multiple of 3,** **and hence** \( a \) **is a multiple of 3. Finally,** \( b = (a + b) - a \) **is a multiple of 3,** **as required.**

2. **In the figure,** \( P \) **is a point on the circumcircle of** \( \triangle ABC \). **Lines** \( AX \) **and** \( CY \) **are drawn so that** \( \angle PAC \neq \angle BAX \) **and** \( \angle PCA \neq \angle BCY \). **Prove that** \( AX \) **and** \( CY \) **are parallel.**

**SOLUTION.** Extend \( BC \) **to meet** \( AX \) **at point** \( D \). **Since** \( \angle ABC \) **is the exterior angle of** \( \triangle ABD \) **at point** \( B \), **it is equal to the sum of the two remote interior angles of this triangle,** **and we have** \( \angle ADB + \angle DAB = \angle ABC \). **Also,** **points** \( B \) **and** \( P \) **are opposite vertices of quadrilateral** \( ABPC \), **which is inscribed in a circle,** **and it follows that** \( \angle ABC = 180^\circ - \angle P \). **Working in** \( \triangle APC \), **we see that** \( 180^\circ - \angle P = \angle PAC + \angle PCA \). **Combining these equalities,** **we get**

\[
\angle ADB + \angle DAB = \angle ABC = 180^\circ - \angle P = \angle PAC + \angle PCA = \angle DAB + \angle DCY,
\]

where the last equality follows since \( \angle PAC = \angle DAB \) **and** \( \angle PCA = \angle DCY \) **by hypothesis. Subtracting** \( \angle DAB \) **from both sides of this equation yields** \( \angle ADB = \angle DCY \), **and hence** \( AX \) **and** \( CY \) **are parallel because they form equal alternate interior angles with the transversal** \( DC \).

3. **(NEW YEAR’S PROBLEM)**

Let us write \( P(n) \) **to denote the smallest prime number that does NOT divide** \( n \) **and use** \( Q(n) \) **to denote the product of all prime numbers less than** \( n \), **with** \( Q(2) \) **defined to be 1. Construct a sequence of numbers** \( X_n \) **as follows. Put** \( X_0 = 1 \) **and for each integer** \( n > 0 \), **define** \( X_n = X_{n-1} P(X_{n-1})/Q(P(X_{n-1})) \). Thus the first several numbers in this sequence are** \( 1, 2, 3, 6, 5, 10, 15, 30, 7, \ldots \). **Compute** \( X_{1998} \).
SOLUTION. Label the primes in increasing order, so that \( p_0 = 2, p_1 = 3, p_2 = 5, p_3 = 7 \), and so on. Given a number \( n \) with binary expansion \( a_s \ldots a_2a_1a_0 \), define \( Y_n \) to be the number obtained by multiplying all primes \( p_i \) for which \( a_i = 1 \). For example, if \( n = 26 \), then the binary expansion of \( n \) is 11010, and so the nonzero digits are \( a_1, a_3 \) and \( a_4 \). Thus \( Y_{26} = p_1p_3p_4 = 3 \cdot 7 \cdot 11 = 231 \).

We will show that \( Y_n = X_n \) for all \( n \geq 1 \). Of course, this is easy to check for the first few values of \( n \). We investigate how to compute \( Y_n \) from \( Y_{n-1} \). To start with, suppose that the binary expansion of \( n - 1 \) is \( a_s \ldots a_2a_1a_0 \) and that the rightmost 0 digit is \( a_k \). Then to obtain the binary expansion of \( n = (n - 1) + 1 \), we change \( a_k \) to 1 and we change \( a_0, a_1, \ldots, a_{k-1} \) to 0. Thus \( Y_n = Y_{n-1}p_k/(p_0p_1 \cdots p_{k-1}) \). But, since \( p_0, p_1, \ldots, p_{k-1} \) all divide \( Y_{n-1} \), we see that \( p_k = P(Y_{n-1}) \) and \( p_0p_1 \cdots p_{k-1} = Q(p_k) = Q(P(Y_{n-1})) \). Hence \( Y_n = Y_{n-1}P(Y_{n-1})/Q(P(Y_{n-1})) \) and this is identical to the rule for getting \( X_n \) from \( X_{n-1} \). It follows that \( X_n = Y_n \) for all \( n \).

Since 1998 = 2^{10} + 2^9 + 2^8 + 2^7 + 2^6 + 2^3 + 2^2 + 2^1, we have \( X_{1998} = Y_{1998} = p_10 \cdot p_9 \cdot p_8 \cdot p_7 \cdot p_6 \cdot p_3 \cdot p_2 \cdot p_1 = 31 \cdot 29 \cdot 23 \cdot 19 \cdot 17 \cdot 7 \cdot 5 \cdot 3 = 701, 260, 455 \).

4. Prove that the average of the squares of three real numbers can never be less than the square of the average of these numbers.

SOLUTION. Let \( a, b \) and \( c \) be the three numbers. The average of the squares of these numbers is \( (a^2 + b^2 + c^2)/3 \) while the square of their average is \( ((a + b + c)/3)^2 \). We are asked to show that \( (a^2 + b^2 + c^2)/3 \geq ((a + b + c)/3)^2 \), and by multiplying both sides by 9, we see that it suffices to show that \( 3(a^2 + b^2 + c^2) \geq (a + b + c)^2 \). Since the right side of this is equal to \( a^2 + b^2 + c^2 + 2(ab + ac + bc) \), it suffices to show that \( 2(a^2 + b^2 + c^2) \geq 2(ab + ac + bc) \).

Observe that \( a^2 + b^2 - 2ab = (a + b)^2 \geq 0 \), and thus \( a^2 + b^2 \geq 2ab \). Similarly, \( a^2 + c^2 \geq 2ac \) and \( b^2 + c^2 \geq 2bc \). Adding these three inequalities yields \( 2(a^2 + b^2 + c^2) \geq 2(ab + ac + bc) \), as we wanted.

5. Find all polynomial functions \( F(x) \) such that \( F(0) = 2 \) and \( F(x^2 + 1) = F(x)^2 + 1 \) for all \( x \).

SOLUTION. Since \( F(0) = 2 \), we can substitute \( x = 0 \) into the equation \( F(x^2 + 1) = F(x)^2 + 1 \) to obtain \( F(1) = 2^2 + 1 = 5 \). Similarly, if we plug \( x = 1 \) into the equation we obtain \( F(2) = 26 \). Next, setting \( x = 2 \), we see that \( F(5) = 26^2 + 1 = 677 \). Continuing in this manner, we can determine the values of \( F(x) \) when \( x \) is any one of the numbers \( 0, 1, 2, 5, 26, 677, \ldots \), where each number in this sequence is one more than the square of the previous number.

We saw that the values of \( F(x) \) on \( 0, 1, 2, 5, \ldots \) are \( 2, 5, 26, 677, \ldots \) respectively, and in general, on this infinite sequence of numbers, the values of \( F(x) \) form exactly the same sequence of numbers, but shifted two places. In other words, if \( n \) is any member of our sequence, then \( F(n) \) is the number that occurs two places later in the sequence. The number following \( n \) is \( n^2 + 1 \), and the number following that is \( (n^2 + 1)^2 + 1 \). In particular, there are infinitely many values of \( x \) for which the polynomial function \( F(x) \) is equal to the polynomial function \( (x^2 + 1)^2 + 1 \). It follows that there are infinitely many values of \( x \) for which the polynomial function \( F(x) - [(x^2 + 1)^2 + 1] \) has the value 0. But a nonzero polynomial of degree \( d \) can have at most \( d \) zeros, so we conclude that \( F(x) - [(x^2 + 1)^2 + 1] \) must actually be identically 0. Consequently, \( F(x) = (x^2 + 1)^2 + 1 = x^4 + 2x^2 + 2 \).