WISCONSIN MATHEMATICS SCIENCE & ENGINEERING TALENT SEARCH

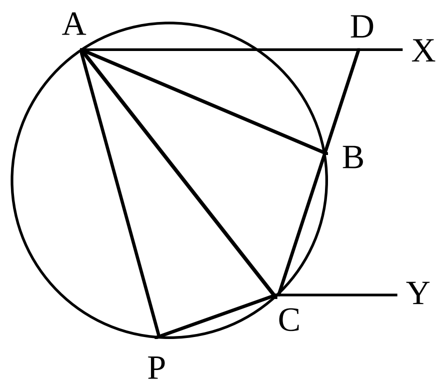
SOLUTIONS TO PROBLEM SET IV (1997-98)

1. Suppose that a and b are integers such that $a + 2b$ and $b + 2a$ are squares. Prove that each of a and b is a multiple of 3.

SOLUTION. Write $a + 2b = m^2$ and $2a + b = n^2$, where m and n are integers. Then $m^2 + n^2 = 3a + 3b$, and so $m^2 + n^2$ is a multiple of 3. We can write m in one of the three forms $m = 3k$, $m = 3k + 1$ or $m = 3k - 1$, where k is some integer. In the first case, m^2 is a multiple of 9, and in the second and third cases, we compute that $m^2 = 9k^2 \pm 6k + 1$, and thus m^2 has the form $3t + 1$ for some integer t. Similarly, of course, n^2 is either a multiple of 9 or else it has the form $3s + 1$ for some integer s. If either of m^2 or n^2 is not a multiple of 9, it therefore follows that $m^2 + n^2$ is either one or two more than a multiple of 3. However, we know that this is not the case, and consequently each of m^2 and n^2 is actually a multiple of 9.

Since $3(a + b) = m^2 + n^2$ is a multiple of 9, we deduce that $a + b$ must be a multiple of 3. Also, $a - b = n^2 - m^2$ is a multiple of 3. (In fact, it is a multiple of 9.) It follows that $2a = (a + b) + (a - b)$ is a multiple of 3, and hence a is a multiple of 3. Finally, $b = (a + b) - a$ is a multiple of 3, as required.

2. In the figure, P is a point on the circumcircle of $\triangle ABC$. Lines $\overline{AX}$ and $\overline{CY}$ are drawn so that $\angle PAC = \angle BAX$ and $\angle PCA = \angle BCY$. Prove that $\overline{AX}$ and $\overline{CY}$ are parallel.



SOLUTION. Extend $\overline{BC}$ to meet $\overline{AX}$ at point D. Since $\angle ABC$ is the exterior angle of $\triangle ABD$ at point B, it is equal to the sum of the two remote interior angles of this triangle, and we have $\angle ADB + \angle DAB = \angle ABC$. Also, points B and P are opposite vertices of quadrilateral $ABCP$, which is inscribed in a circle, and it follows that $\angle ABC = 180^\circ - \angle P$. Working in $\triangle APC$, we see that $180^\circ - \angle P = \angle PAC + \angle PCA$. Combining these equalities, we get

$$\angle ADB + \angle DAB = \angle ABC = 180^\circ - \angle P = \angle PAC + \angle PCA = \angle DAB + \angle DCY,$$

where the last equality follows since $\angle PAC = \angle DAB$ and $\angle PCA = \angle DCY$ by hypothesis. Subtracting $\angle DAB$ from both sides of this equation yields $\angle ADB = \angle DCY$, and hence $\overline{AX}$ and $\overline{CY}$ are parallel because they form equal alternate interior angles with the transversal $\overline{DC}$.

3. (NEW YEAR'S PROBLEM)
Let us write $P(n)$ to denote the smallest prime number that does NOT divide n and use $Q(n)$ to denote the product of all prime numbers less than n, with $Q(2)$ defined to be 1. Construct a sequence of numbers X_n as follows. Put $X_0 = 1$ and for each integer $n > 0$, define $X_n = X_{n-1}P(X_{n-1})/Q(P(X_{n-1}))$. Thus the first several numbers in this sequence are 1, 2, 3, 6, 5, 10, 15, 30, 7, ... Compute X_{1998}.

SOLUTION. Label the primes in increasing order, so that $p_0 = 2$, $p_1 = 3$, $p_2 = 5$, $p_3 = 7$, and so on. Given a number n with binary expansion $a_s \ldots a_2a_1a_0$, define Y_n to be the number obtained by multiplying all primes p_i for which $a_i = 1$. For example, if $n = 26$, then the binary expansion of n is 11010, and so the nonzero digits are a_1, a_3 and a_4. Thus $Y_{26} = p_1p_3p_4 = 3{\cdot}7{\cdot}11 = 231$.

We will show that $Y_n = X_n$ for all $n \geq 1$. Of course, this is easy to check for the first few values of n. We investigate how to compute Y_n from Y_{n-1}. To start with, suppose that the binary expansion of $n-1$ is $a_s \ldots a_2a_1a_0$ and that the rightmost 0 digit is a_k. Then to obtain the binary expansion of $n = (n-1)+1$, we change a_k to 1 and we change $a_0, a_1, \ldots, a_{k-1}$ to 0. Thus $Y_n = Y_{n-1}p_k/(p_0p_1\cdots p_{k-1})$. But, since $p_0, p_1, \ldots, p_{k-1}$ all divide Y_{n-1}, we see that $p_k = P(Y_{n-1})$ and $p_0p_1\cdots p_{k-1} = Q(p_k) = Q(P(Y_{n-1}))$. Hence $Y_n = Y_{n-1}P(Y_{n-1})/Q(P(Y_{n-1}))$ and this is identical to the rule for getting X_n from X_{n-1}. It follows that $X_n = Y_n$ for all n.

Since $1998 = 2^{10} + 2^9 + 2^8 + 2^7 + 2^6 + 2^3 + 2^2 + 2^1$, we have $X_{1998} = Y_{1998} = p_{10}{\cdot}p_9{\cdot}p_8{\cdot}p_7{\cdot}p_6{\cdot}p_3{\cdot}p_2{\cdot}p_1 = 31{\cdot}29{\cdot}23{\cdot}19{\cdot}17{\cdot}7{\cdot}5{\cdot}3 = 701,260,455$.

4. Prove that the average of the squares of three real numbers can never be less than the square of the average of these numbers.

SOLUTION. Let a, b and c be the three numbers. The average of the squares of these numbers is $(a^2+b^2+c^2)/3$ while the square of their average is $((a+b+c)/3)^2$. We are asked to show that $(a^2+b^2+c^2)/3 \geq ((a+b+c)/3)^2$, and by multiplying both sides by 9, we see that it suffices to show that $3(a^2+b^2+c^2) \geq (a+b+c)^2$. Since the right side of this is equal to $a^2+b^2+c^2+2(ab+ac+bc)$, it suffices to show that $2(a^2+b^2+c^2) \geq 2(ab+ac+bc)$.

Observe that $a^2+b^2-2ab = (a+b)^2 \geq 0$, and thus $a^2+b^2 \geq 2ab$. Similarly, $a^2+c^2 \geq 2ac$ and $b^2+c^2 \geq 2bc$. Adding these three inequalities yields $2(a^2+b^2+c^2) \geq 2(ab+ac+bc)$, as we wanted.

5. Find all polynomial functions $F(x)$ such that $F(0) = 2$ and $F(x^2+1) = F(x)^2+1$ for all x.

SOLUTION. Since $F(0) = 2$, we can substitute $x = 0$ into the equation $F(x^2+1) = F(x)^2+1$ to obtain $F(1) = 2^2+1 = 5$. Similarly, if we plug $x = 1$ into the equation we obtain $F(2) = 26$. Next, setting $x = 2$, we see that $F(5) = 26^2+1 = 677$. Continuing in this manner, we can determine the values of $F(x)$ when x is any one of the numbers $0, 1, 2, 5, 26, 677, \ldots$, where each number in this sequence is one more than the square of the previous number.

We saw that the values of $F(x)$ on $0, 1, 2, 5, \ldots$ are $2, 5, 26, 677, \ldots$ respectively, and in general, on this infinite sequence of numbers, the values of $F(x)$ form exactly the same sequence of numbers, but shifted two places. In other words, if n is any member of our sequence, then $F(n)$ is the number that occurs two places later in the sequence. The number following n is n^2+1, and the number following that is $(n^2+1)^2+1$. In particular, there are infinitely many values of x for which the polynomial function $F(x)$ is equal to the polynomial function $(x^2+1)^2+1$. It follows that there are infinitely many values of x for which the polynomial function $F(x)-[(x^2+1)^2+1]$ has the value 0. But a nonzero polynomial of degree d can have at most d zeros, so we conclude that $F(x)-[(x^2+1)^2+1]$ must actually be identically 0. Consequently, $F(x) = (x^2+1)^2+1 = x^4+2x^2+2$.