WISCONSIN MATHEMATICS SCIENCE & ENGINEERING TALENT SEARCH

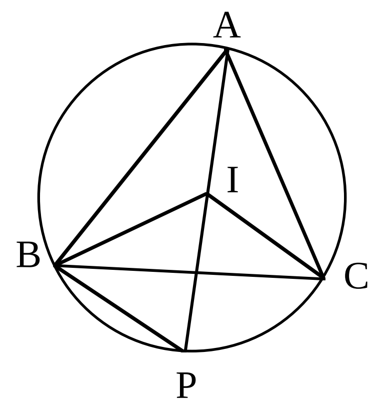
SOLUTIONS TO PROBLEM SET III (1997-98)

1. Given two real numbers a and b, suppose that the average of their fourth powers is equal to the fourth power of their average. Show that the two numbers must be equal.

SOLUTION. We are given that $(a^4 + b^4)/2 = [(a + b)/2]^4$, and multiplication by 16 yields $8(a^4 + b^4) = (a + b)^4$. If both a and b are zero, there is nothing to prove, so we can assume that $a \neq 0$ and we write $x = b/a$. Substitution of $b = ax$ into the above equation yields $8a^4(1 + x^4) = a^4(1 + x)^4$, and thus $8(1 + x^4) = (1 + x)^4$. Our goal is to show that $x = 1$ is the only real solution for this equation.

Subtracting $16x^2$ from both sides, we obtain $8(1 - 2x^2 + x^4) = (1 + x)^4 - 16x^2$. Both sides can now be factored, and we have $8(x - 1)^2(x + 1)^2 = [(1 + x)^2 - 4x][(1 + x)^2 + 4x]$. Observe that the first factor on the right equals $(x - 1)^2$. Since we are seeking solutions (if any) with $x \neq 1$, we can assume that $(x - 1)^2 \neq 0$, and we can then divide both sides by this quantity. We obtain $8(x + 1)^2 = (1 + x)^2 + 4x$, and simplification of this equation now yields $7x^2 + 10x + 7 = 0$. This quadratic equation has no real solutions since its "discriminant" is $10^2 - 4{\cdot}7{\cdot}7$, which is negative. The original equation thus has no real solution different from 1.

2. Let I be the center of the inscribed circle of $\triangle ABC$. Show that the center of the circumscribed circle of $\triangle BIC$ lies on the circumscribed circle of $\triangle ABC$.



SOLUTION. Draw line $\overline{AI}$ and extend it to meet the circumcircle of $\triangle ABC$ at point P, as shown in the diagram. We will show that P is the center of the circumcircle of $\triangle BIC$ by proving that $PB = PI = PC$.

To prove that $PB = PI$, it suffices to show that $\angle PBI = \angle PIB$, and so we compute these two angles. First, $\angle PIB$ is the exterior angle at point I of $\triangle ABI$, and hence it is equal to the sum of the interior angles at the other two points of this triangle. Therefore $\angle PIB = \angle IAB + \angle IBA$. Recall, however, that the "incenter" I lies on the angle bisectors of the three angles of the original triangle, and thus $\angle IAB = \frac{1}{2}\angle A$ and $\angle IBA = \frac{1}{2}\angle B$. Consequently, $\angle PIB = \frac{1}{2}\angle A + \frac{1}{2}\angle B$.

Next, observe that $\angle PBI = \angle PBC + \angle CBI = \angle PBC + \frac{1}{2}\angle B$. But $\angle PBC$ is inscribed in the circumcircle of $\triangle ABC$, and it subtends the same arc as the inscribed angle $\angle PAC$. It follows that $\angle PBC = \angle PAC = \frac{1}{2}\angle A$. Thus $\angle PIB = \frac{1}{2}\angle A + \frac{1}{2}\angle B = \angle PBI$, and it follows that $PB = PI$, as desired. A similar argument shows that $PC = PI$ and the proof is complete.

3. Recall that a rational number is one that can be written in the form m/n, where m and n are integers. Suppose that a, b and c are positive rational numbers and that $\sqrt{a} + \sqrt{b} + \sqrt{c}$ is also rational. Show that $\sqrt{a}$, $\sqrt{b}$ and $\sqrt{c}$ are each rational.

SOLUTION. Note that the set of rational numbers is closed under addition, subtraction, multiplication and division (except that we cannot divide by 0, of course). Write $\sqrt{a} + \sqrt{b} + \sqrt{c} = r$, where r is rational, and note that $r > 0$. Then $\sqrt{a} + \sqrt{b} = r - \sqrt{c}$, and by squaring both sides we get $a + 2\sqrt{a}\sqrt{b} + b = r^2 - 2r\sqrt{c} + c$. We can thus write $\sqrt{ab} = s - r\sqrt{c}$, where $s = (r^2 + c - a - b)/2$, and we observe that $s \neq 0$ since the positive number $\sqrt{ab}$ is definitely not equal to the negative number $-r\sqrt{c}$. Also, s is rational because of the closure properties of the rational numbers. (We are using the fact that each of a, b, c and r is rational.) Squaring both sides of the equation $\sqrt{ab} = s - r\sqrt{c}$, we get $ab = s^2 - 2rs\sqrt{c} + r^2c$, and thus $\sqrt{c} = (s^2 + r^2c - ab)/2rs$ is rational because of the closure properties of the rationals. (Note that it is legal to divide by $2rs$ since we know that $s \neq 0$ and $r \neq 0$.) We have shown that $\sqrt{c}$ is rational, and similar arguments show that $\sqrt{a}$ and $\sqrt{b}$ are also rational.

4. Suppose that all the integers $n > 1000$ are divided into two sets A and B. Show that at least one of these sets contains two different numbers x and y such that $x + y$ is also in that set.

SOLUTION. At least one of A or B must be an infinite set, and so we can assume that A is infinite. Choose two different members of A, say u and v. Since A is infinite, we can then choose w in A so large that that $w > 2u + v + 1000$. If either $v + u$ or $v + w$ is in A, we are done, and so we can assume that both of these numbers lie in B.

Consider the number $w - u$. Because $w > 2u + v + 1000$, we know that $w - u > 1000$ and also that $w - u > u$ and $w - u > u + v$. Now $w - u$ must lie in one of the two sets, and we suppose first that $w - u$ is in A. Then $w - u$ and u are two different members of A, and $(w - u) + u = w$, which is also in A, and so we are done in this case. On the other hand, if $w - u$ is in B, then $w - u$ and $u + v$ are different members of B and $(w - u) + (u + v) = w + v$, which also lies in B. We are thus done in this case too.

5. Recall that the Fibonacci numbers are $1, 1, 2, 3, 5, 8, \ldots$, where after the first two, each is the sum of the preceding two numbers. Write F_n to denote the nth Fibonnaci number and let r denote the number $(1 + \sqrt{5})/2$. Prove that the ratio F_{100}/F_{99} is so close to r that the difference satisfies $|F_{100}/F_{99} - r| < 10^{-20}$.

SOLUTION. First, note that the number r is a solution of the quadratic equation $x^2 - x - 1 = 0$, and thus $r^2 = r + 1$. Now write $D_n = F_n/F_{n-1} - r$, so our task is to show that $|D_{100}| < 10^{-20}$. We investigate what happens to the quantity D_n when we increase n by 1. We have

$$D_{n+1} = \frac{F_{n+1}}{F_n} - r = \frac{F_n + F_{n-1}}{F_n} - r = 1 - r + \frac{F_{n-1}}{F_n} = 1 - r + \frac{1}{D_n + r}\,.$$

Simplifying this by elementary algebra, and using the fact that $r^2 = r + 1$, we deduce that

$$D_{n+1} = \frac{1 - r}{D_n + r} D_n\,.$$

Next, we observe that $D_n + r = F_n/F_{n-1} > 1$ since the Fibonacci numbers form an increasing sequence. It follows that $|(1 - r)/(D_n + r)| < |1 - r|$, and therefore $|D_{n+1}| < |1 - r||D_n|$. Since $D_2 = 1 - r$, this yields $|D_n| < |1 - r|^{n-1}$ for $n > 2$, and in particular, $|D_{100}| < |1 - r|^{99}$. A calculator computation shows that $|1 - r|^{99}$ is about 2.04×10^{-21}, and this is less than 10^{-20}.