WISCONSIN MATHEMATICS SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET I (1997-98)

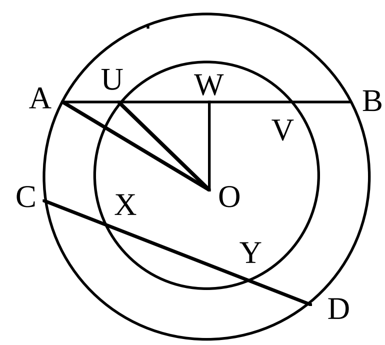
1. Everyone has just one "magic birthday", when his age is exactly equal to the sum of the digits of the year of his birth. For example, the magic birthday of someone born in 1899 was in 1926. Notice that someone born in 1908 also had a magic birthday in 1926. What is the next year after 1926 in which two people born in different years can both have magic birthdays.

SOLUTION. The digit sum of 1982 is 20, and thus the magic birthday of someone born in 1982 will occur in 2002. Furthermore, a person born in the year 2000 will also have a magic birthday in 2002. Thus 2002 is one of those unusual years in which two people born in different years have a magic birthday. We will show that 2002 is the first time after 1926 when this happens.

The digit sum of every year before 1899 is at most 27, and so the magic birthday of a person born before 1899 was earlier than 1926. A person whose magic birthday occurs later than 1926, therefore, must be born in the year 1900 or later. For birth year $19ab = 1900+10a+b$, the digit sum is $10+a+b$, so the magic birthday occurs in year $1900+10a+b+10+a+b = 1910+11a+2b$. Similarly, for birth year $19cd$, the magic birthday occurs in year $1910 + 11c + 2d$. If these two magic birthdays occur in the same year, we have $1910 + 11a + 2b = 1910 + 11c + 2d$, and it follows that $11(a-c) = 2(d-b)$. This forces $d-b$ to be a multiple of 11. But d and b are digits, so $d-b$ lies between -9 and 9, and hence the only possibility is that $d-b=0$, and therefore $a-c=0$. Consequently, $d=b$, $a=c$, and we have shown that if two people born in years of the form $19xx$ have magic birthdays in the same year, they must have been born in the same year.

If two people born in different years have magic birthdays in the same year, and that year is later than 1926, we have shown that at least one of the people was born later than 1999, and thus that person's magic birthday year cannot be earlier than 2002. This is what we wanted to prove.

2. Suppose chords $\overline{AB}$ and $\overline{CD}$ of a circle meet a smaller concentric circle at points U, V, X and Y, as shown. If $AU = 2$, $UV = 10$ and $CX = 3$, find XY and prove that your answer is correct.



SOLUTION. Let $r > s$ be the radii of the two circles and let O be their common center. Draw the line $\overline{OW}$ perpendicular to $\overline{AB}$, and draw lines $\overline{AO}$ and $\overline{UO}$. Then we know that $AO = r$, $UO = s$, and that W bisects the chords $\overline{AB}$ and $\overline{UV}$. In particular, if $UW = y$, then $UV = 2y$. Set $AU = x$ and $WO = z$. Then we conclude from right triangles $\triangle AWO$ and $\triangle UWO$ that

$$r^2 = (x+y)^2 + z^2 \qquad \text{and} \qquad s^2 = y^2 + z^2.$$

Subtracting yields

$$r^2 - s^2 = x^2 + 2xy = x(x+2y) = AU \cdot AV.$$

Similarly, we have $CX \cdot CY = r^2 - s^2 = AU \cdot AV$. Since $AU = 2$, $AV = 2 + 10 = 12$, and $CX = 3$, we conclude that $CY = 8$ and hence that $XY = 8 - 3 = 5$.

3. Recall that for each positive integer n, we write $n!$ to denote the product $1\cdot 2\cdot 3\cdot \ldots \cdot n$. (This is called "$n$ factorial".) Prove that $n!$ can never be a multiple of 2^n for any positive integer n.

SOLUTION. Suppose there is some positive integer n such that 2^n divides $n!$. There must be a smallest such integer, and so we can assume that 2^n divides $n!$, but that 2^m does not divide $m!$ for $1 \le m < n$. (We will work to derive a contradiction from this assumption.)

We can factor $n! = ab$, where a is the product of all the odd numbers in the range from 1 to n and b is the product of the even numbers in this range. Then a is an odd number, and

$$b = 2\cdot 4\cdot 6\cdot \cdots \cdot 2(e-1)\cdot 2e\,,$$

where $2e$ is the largest even number with $2e \le n$. Thus $b = 2^e\cdot e!$ and we have $n = a\cdot 2^e\cdot e!$. We are assuming that 2^n divides $n!$, and since a is odd, it follows that 2^n divides $2^e\cdot e!$, and thus 2^{n-e} divides $e!$. But $e < n$, so we know that 2^e does *not* divide $e!$, and therefore we must have $n-e < e$. Thus $n < 2e$, and this is a contradiction, since we know that $2e \le n$. This contradiction arose from our assumption that there was some positive integer n such that 2^n divides $n!$, and consequently we can conclude that no such integer n exists.

4. Let $a < b < c < d < e$ be real numbers and let S be the set of all possible sums obtained by adding two distinct numbers from these five. If S has only seven members, show that a, b, c, d and e form an arithmetic progression.

SOLUTION. Since $a < b < c < d < e$, we clearly have

$$a+b < a+c < a+d < a+e < b+e < c+e < d+e\,,$$

and so these seven distinct sums exhaust the set S. We have $a+c < b+c < b+d < b+e$, and thus $b+c$ and $b+d$ are two members of the set S that lie between $a+c$ and $b+e$. We see, however, that the only members of S that lie between $a+c$ and $b+e$ are $a+d$ and $a+e$. We deduce, therefore, that $b+c = a+d$ and that $b+d = a+e$, and thus $b-a = d-c$, and also $b-a = e-d$.

To show that a, b, c, d and e form an arithmetic progression, we need to prove that $b-a = c-b = d-c = e-d$, and we now know that three of these four differences are equal. It now remains to show that $c-b$ is equal to the other three differences. For this, observe that $a+e = b+d < c+d < c+e$, and thus $c+d$ lies between $a+e$ and $c+e$ in the set S. It follows that $c+d = b+e$, and therefore $c-b = e-d$, as required.

5. Let x and y be positive real numbers satisfying $x^3 + y^3 = 2xy$. Show that $x < 2^{2/3}$ and $y < 2^{2/3}$. (This is, in fact, not the best possible bound.)

SOLUTION. By the symmetry of the problem, it suffices to assume that $x \ge y$, and therefore $2xy = x^3+y^3 \ge 2y^3$. Dividing by $2y$, we deduce that $x \ge y^2$. We also have $2xy = x^3+y^3 > x^3$, since $y > 0$, and division by $2x$ yields $y > x^2/2$. Combining the two inequalities we have already obtained, we conclude that $x \ge y^2 > (x^2/2)^2 = x^4/4$, and hence that $4 > x^3$. Thus $y \le x < 4^{1/3} = 2^{2/3}$, as desired.