WISCONSIN MATHEMATICS SCIENCE & ENGINEERING TALENT SEARCH

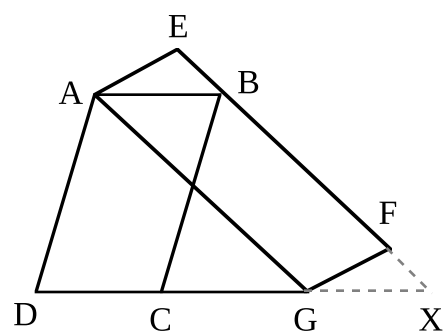
SOLUTIONS TO PROBLEM SET V (1996-97)

1. Let $\{a_1, a_2, \ldots, a_{100}\}$ be a set of 100 different real numbers and assume that $a_i > i$ for each subscript i. Now let b_1 be the smallest of the numbers a_i, let b_2 be the second smallest, and so on, so that $\{b_1, b_2, \ldots, b_{100}\}$ is the original set of numbers sorted in such a way that $b_1 < b_2 < \cdots < b_{100}$. Show that $b_i > i$.

SOLUTION. Fix an integer i with $1 \le i \le 100$. Let us first consider how many members of our set $A = \{a_1, a_2, \ldots, a_{100}\}$ can fail to exceed i. If $j \ge i$, then we know that $a_j > j \ge i$ and thus $a_j > i$. Consequently, $a_1, a_2, \ldots, a_{i-1}$ are the only possible members of A not greater than i, and hence fewer than i members of A can fail to exceed i.

Now suppose, by way of contradiction, that $b_i \le i$. Since $b_1 < b_2 < \cdots < b_i \le i$ it follows that there are at least i members of A not exceeding i. But this contradicts our previous observation that at most $i - 1$ members of A are $\le i$, and thus $b_i > i$, as required.

2. In the figure, $ABCD$ and $AEFG$ are parallelograms, where B lies on line $\overline{EF}$ and C lies on $\overline{DG}$. Show that the two parallelograms have equal areas.

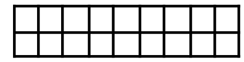


SOLUTION. Extend lines $\overline{EF}$ and $\overline{DG}$ to meet at X, as shown. Now $\overline{AB}$ is parallel to $\overline{GX}$, since $\overline{AB}$ is parallel to $\overline{DC}$. Also, $\overline{BX}$ is parallel to $\overline{AG}$, and thus $ABXG$ is a parallelogram. View $\overline{AB}$ as the base of each of the two parallelograms $ABCD$ and $ABXG$, and note that the heights of these two parallelograms are equal, since these heights are both equal to the perpendicular distance from A to $\overline{DX}$. It follows that $ABCD$ and $ABXG$ have equal areas. Now view $\overline{AG}$ as the base of the parallelograms $AEFG$ and $ABXG$. Since these parallelograms both have height equal to the distance from G to $\overline{EX}$, it follows that they also have equal areas. With this, we conclude that the areas of $ABCD$, $ABXG$ and $AEFG$ are all equal.

3. Let $\{x\}$ denote the fractional part of the real number x so that, for example, $\{12/5\} = 2/5$ and $\{3\} = 0$. Find the smallest number x, larger than 1, with $\{x\} + \{1/x\} = 1$.

SOLUTION. Let x be a real number larger than 1 satisfying $\{x\} + \{1/x\} = 1$. If n denotes the greatest integer in x, then $x = n + \{x\}$ and, of course, $n \ge 1$. Since $0 < 1/x < 1$, we have $\{1/x\} = 1/x$. Thus $x + 1/x = n + \{x\} + \{1/x\} = n + 1$, and we see that x is a solution of the quadratic equation $x^2 - (n+1)x + 1 = 0$. In particular, the quadratic formula implies that $x = (1/2)\big[(n+1) \pm \sqrt{(n+1)^2 - 4}\big]$, and since $x > 1$, we can discard the solution with the minus sign. In other words, $x = (1/2)\big[(n+1) + \sqrt{(n+1)^2 - 4}\big]$. Now, if $n = 1$, then $x = 1$, a contradiction. Thus $n \ge 2$ and the potentially smallest x occurs when $n = 2$ and $x = (3 + \sqrt{5})/2$. To check that this number actually satisfies the appropriate condition, note that $x \approx 2.62$, so $\{x\} = x - 2 = (-1 + \sqrt{5})/2$. Furthermore, $\{1/x\} = 1/x = 2/(3 + \sqrt{5}) = (3 - \sqrt{5})/2$, and consequently $\{x\} + \{1/x\} = (-1 + \sqrt{5})/2 + (3 - \sqrt{5})/2 = 1$, as required.

4. I have a 2 by 9 grid, as indicated, and in each box, I want to write one of the numbers 1, 2, . . . , 9 in such a way that each of these numbers appears twice. I also require that the two occurrences of each number are in boxes that are either horizontally or vertically adjacent. In how many ways can this be done?



SOLUTION. Imagine that we have a set of dominoes, each exactly the right size to cover two adjacent boxes of the grid. For each way of filling in numbers in the appropriate manner, we can cover the grid with dominoes in such a way that each domino lies over two boxes containing the same number. This results in a complete covering of the grid with 9 dominoes.

Let us consider how many different numberings correspond to each domino covering. Note that the number 1 lies under the two halves of any one of the 9 dominoes; the number 2 lies under any one of the remaining 8 dominoes; the number 3 lies under the two halves of any one of the remaining 7 dominoes, and so on. Therefore, for each such covering by dominoes, we obtain $9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 362,880$ different numberings. To solve the problem, therefore, we must multiply the number of possible domino coverings by this large number (equal to 9!).

Our task now is to count the possible domino coverings of a 2×9 grid. To this end, let a_n denote the number of domino coverings of a $2 \times n$ grid, so that what we want is a_9. Note that $a_1 = 1$ and $a_2 = 2$. In general, to compute a_n for $n \geq 3$, look at the left end of a covered $2 \times n$ grid. There are two possibilities: either a single vertical domino will cover the leftmost two boxes, or else two horizontal dominoes will cover the leftmost four boxes. In the first case, there are a_{n-1} ways to finish, since the remaining boxes form a $2 \times (n-1)$ grid. In the second case, there are a_{n-2} ways to finish. We deduce that $a_n = a_{n-1} + a_{n-2}$ when $n \geq 3$.

Since $a_1 = 1$ and $a_2 = 2$, we see that $a_3 = 3$, $a_4 = 5$, $a_5 = 8$, $a_6 = 13$, $a_7 = 21$, a_8=34 and finally $a_9 = 55$. There are thus $a_9 \cdot 9! = 55 \cdot (362,880) = 19,958,400$ ways to fill in the numbers.

5. I compute $1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \cdots$, where at the nth step I add the number $\frac{1}{n^3}$. Show that no matter how long I keep this up, my sum will never exceed 1.25.

SOLUTION. If $n \geq 2$, then $n^3 > n(n^2-1) = (n-1)n(n+1) > 0$, so

$$\frac{1}{n^3} < \frac{1}{(n-1)n(n+1)} = \frac{1}{2}\left[\frac{1}{(n-1)n} - \frac{1}{n(n+1)}\right].$$

Thus, if we let s_k denote the sum $1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \cdots + \frac{1}{k^3}$, then

$$s_k < 1 + \frac{1}{2}\left[\left(\frac{1}{1 \cdot 2} - \frac{1}{2 \cdot 3}\right) + \left(\frac{1}{2 \cdot 3} - \frac{1}{3 \cdot 4}\right) + \cdots + \left(\frac{1}{(k-1)k} - \frac{1}{k(k+1)}\right)\right].$$

Note that the terms $\pm\frac{1}{2 \cdot 3}, \pm\frac{1}{3 \cdot 4}, \ldots, \pm\frac{1}{(k-1)k}$ cancel in pairs and therefore

$$s_k < 1 + \frac{1}{2}\left[\frac{1}{1 \cdot 2} - \frac{1}{k(k+1)}\right] < 1 + \frac{1}{4} = 1.25,$$

as required. By using a computer or calculator, we can verify that $s_{16} > 1.20$ and that s_k is approximately equal to 1.202056903 for very large values of k. No one knows the precise value of this infinite sum.