WISCONSIN MATHEMATICS SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET III (1996-97)

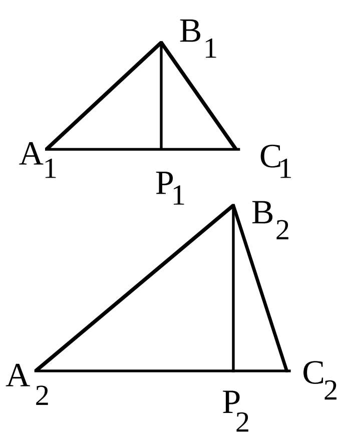
1. Let $\square$ be a binary operation defined on the set of real numbers. (This means that if x and y are any two real numbers, then $x \square y$ is a real number determined by x and y.) Suppose that $(a \square b) + c = (b + c) \square (a + c)$ and $0 \square (a + b) = (0 \square a) + (0 \square b)$ for all real numbers a, b and c. Compute $23 \square 77$ and prove that your answer is correct.

SOLUTION. First, put $a = b = 0$ in the second equation. This yields $(0 \square 0) + (0 \square 0) = 0 \square (0 + 0) = 0 \square 0$, so $0 \square 0 = 0$. Next, if we set $a = -b$ in the second equation, we obtain $(0 \square (-b)) + (0 \square b) = 0 \square ((-b) + b) = 0 \square 0 = 0$, so $0 \square (-b) = -(0 \square b)$. Now, we use the first equation and let $a = 0$ and $c = -b$. This yields $(0 \square b) - b = (b + (-b)) \square (0 + (-b)) = 0 \square (-b) = -(0 \square b)$. Hence $2(0 \square b) = b$ and $0 \square b = b/2$. Finally, if we put $c = -b$ in the first equation, then $(a \square b) - b = (b + (-b)) \square (a + (-b)) = 0 \square (a - b) = (a - b)/2$. Thus $a \square b = b + (a - b)/2 = (a + b)/2$ and $23 \square 77 = (23 + 77)/2 = 50$.

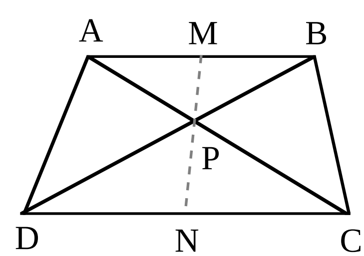
2. The diagonals of quadrilateral $ABCD$ meet at point P, as shown. When the midpoints M of $\overline{AB}$ and N of $\overline{DC}$ are joined, it is found that the line $\overline{MN}$ goes through the point P. Prove that $\overline{AB}$ and $\overline{DC}$ are parallel.

SOLUTION. We begin by establishing a useful general fact. Let $\triangle A_1B_1C_1$ and $\triangle A_2B_2C_2$ have areas K_1 and K_2 respectively, and suppose that $\angle A_1 = \angle A_2$. Then

$$\frac{K_1}{K_2} = \frac{A_1B_1 \cdot A_1C_1}{A_2B_2 \cdot A_2C_2}.$$



To prove this, draw altitudes $\overline{B_1P_1}$ and $\overline{B_2P_2}$, and note that $K_1 = \frac{1}{2}B_1P_1 \cdot A_1C_1$ and $K_2 = \frac{1}{2}B_2P_2 \cdot A_2C_2$. Consequently, we have $K_1/K_2 = (B_1P_1/B_2P_2)(A_1C_1/A_2C_2)$. Also, $\triangle A_1B_1P_1$ is similar to $\triangle A_2B_2P_2$ since $\angle A_1 = \angle A_2$ and $\angle A_1P_1B_1 = 90^\circ = \angle A_2P_2B_2$. It follows that $B_1P_1/B_2P_2 = A_1B_1/A_2B_2$ and thus $K_1/K_2 = (B_1P_1/B_2P_2)(A_1C_1/A_2C_2) = (A_1B_1/A_2B_2)(A_1C_1/A_2C_2)$.



In the situation of Problem 2, we observe that the areas of $\triangle APM$ and $\triangle BPM$ are equal since they have the same base $\overline{AM} = \overline{BM}$ and the same altitude to P. Similarly, the areas of $\triangle DPN$ and $\triangle CPN$ are equal. Since $\angle BPM = \angle DPN$ and $\angle APM = \angle CPN$, it follows from the general fact proved above that

$$\frac{PB \cdot PM}{PD \cdot PN} = \frac{\text{area}(\triangle PMB)}{\text{area}(\triangle PND)} = \frac{\text{area}(\triangle PMA)}{\text{area}(\triangle PNC)} = \frac{PA \cdot PM}{PC \cdot PN}.$$

Canceling PM/PN from both sides, we get $PB/PD = PA/PC$. Also $\angle APB = \angle CPD$, and therefore we conclude that $\triangle APB$ is similar to $\triangle CPD$. Thus $\angle PAB = \angle PCD$, and hence $\overline{AB}$ and $\overline{DC}$ are parallel, as desired.

3. Let a, b and c be positive real numbers and suppose that $a^m + b^m = c^m$ and $a^n + b^n = c^n$, where m and n are also positive real numbers. Show that $m = n$.

SOLUTION. Suppose that $n > m > 0$. Now $a^m + b^m = c^m$ implies that $0 < a < c$ and $0 < b < c$. Consequently, $0 < a^{n-m} < c^{n-m}$ and $0 < b^{n-m} < c^{n-m}$, and hence

$$\begin{aligned} a^n + b^n = c^n = c^m c^{n-m} &= (a^m + b^m)c^{n-m} = a^m c^{n-m} + b^m c^{n-m} \\ &> a^m a^{n-m} + b^m b^{n-m} = a^n + b^n. \end{aligned}$$

This is a contradiction, and therefore the inequality $n > m > 0$ cannot be true.

4. I have a magic money machine that does the following. When I put in a nickel, the machine gives back two dimes. When I put in a dime, out pop a quarter and three nickels. Inserting a quarter yields a one dollar coin and finally, if I put in a dollar coin, the machine returns three dimes and two nickels. Starting with just one dollar coin, and using the money machine repeatedly, is it possible to end up with coins worth exactly \$50?

SOLUTION. We can never have exactly \$50.00 if we start with \$1.00 no matter which coins we insert in the money machine. To see why this is so, we observe that if we insert a nickel, we receive 20¢, for a net gain of 15¢. If we insert a dime, we get back 40¢ for a gain of 30¢. Similarly, the insertion of a quarter yields a gain of 75¢, while the insertion of a dollar results in a loss of 60¢. In all cases, therefore, the machine changes the value of the money we have by a multiple of 15¢. Consequently, if we use the machine repeatedly, the change in the total value of our money will also be an integer multiple of 15¢. But \$50.00 − \$1.00 = 4900¢ is not an integer multiple of 15¢ and therefore we cannot reach exactly \$50.00 if we start with \$1.00.

5. One solution for the equation $a^2 + b^2 + c^2 + 2 = abc$ is $a = 3$, $b = 3$ and $c = 4$. Do there exist integer solutions of this equation with a, b and c all larger than 10? Either find such a solution or prove that none exists.

SOLUTION. Suppose we are given two positive integers a and b, and we wish to find numbers x such that $a^2 + b^2 + x^2 + 2 = abx$. Then we are really trying to solve the quadratic equation $x^2 - abx + (a^2 + b^2 + 2) = 0$. Of course, there may not be any integer solutions here. But, if we know that there is some integer solution $x = c$, then there must exist a real number d with

$$(*) \qquad x^2 - abx + (a^2 + b^2 + 2) = (x - c)(x - d).$$

Indeed, by comparing the coefficients of x on both sides of equation $(*)$, we see that $ab = c + d$, so $d = ab - c$ is also an integer. Now, $x = d$ will certainly be another solution to the problem with the same values for a and b. In other words, given any three integers a, b and c such that $a^2 + b^2 + c^2 + 2 = abc$, we can replace c by $ab - c$ to get another solution. For example, starting with the given solution $(4, 3, 3)$, we can replace one of the 3's by $3 \cdot 4 - 3 = 9$ to get the new solution $(4, 3, 9)$. Finally, to get a solution where all three numbers exceed 10, we replace the smallest number (which we call c) at each step by the number $ab - c$. In particular, if we list the numbers at each step in decreasing order, then we find the following solutions in turn:

$$(4, 3, 3) \to (9, 4, 3) \to (33, 9, 4) \to (293, 33, 9) \to (9660, 293, 33)$$

Thus $a = 9660$, $b = 293$ and $c = 33$ is an appropriate solution.