WISCONSIN MATHEMATICS SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET I (1996-97)

1. Given real numbers $x,\ a,\ b$ with $x \geq a \geq b \geq 0$, show that

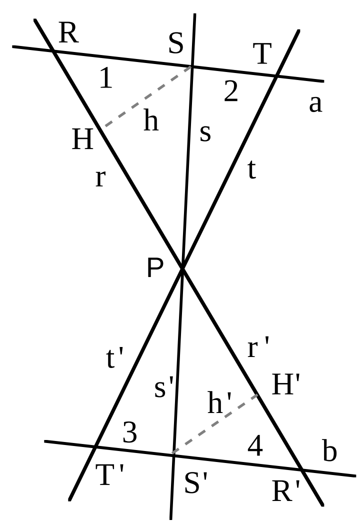
$$\sqrt{x+b} - \sqrt{x-a} \geq \sqrt{x+a} - \sqrt{x-b}$$

SOLUTION. Suppose, by way of contradiction, that $\sqrt{x+a} - \sqrt{x-b} > \sqrt{x+b} - \sqrt{x-a}$. Then $\sqrt{x+a} + \sqrt{x-a} > \sqrt{x+b} + \sqrt{x-b}$. Since the latter quantities are nonnegative, we can square both sides of the inequality to get

$$(x+a) + 2\sqrt{(x+a)(x-a)} + (x-a) > (x+b) + 2\sqrt{(x+b)(x-b)} + (x-b).$$

Simplification of this yields $\sqrt{x^2-a^2} > \sqrt{x^2-b^2}$. Again, squaring both sides and simplifying, we get $b^2 > a^2$, which is not true. This contradiction proves the stated inequality.

2. In the figure, we see three lines through a common point P. These are cut by the two lines a and b as shown, creating four triangular regions, labeled 1, 2, 3 and 4 in the diagram. If the areas of regions 1 and 4 are equal and the areas of regions 2 and 3 are equal, prove that lines a and b are parallel.



SOLUTION. Let $R,\ S,\ T,\ R',\ S',\ T'$ be the six intersection points, as indicated on the figure, and let $r,\ s,\ t,\ r',\ s',\ t'$ denote the lengths of the line segments joining these points to P. In addition, let SH be perpendicular to RP and have length h, and let $S'H'$ be perpendicular to $R'P$ and have length h'. Since $\triangle RPS$ and $\triangle R'PS'$ have the same area, and since SH and $S'H'$ are altitudes of these respective triangles, we see that

$$hr/2 = \text{Area of } \triangle RPS = \text{Area of } \triangle R'PS' = h'r'/2.$$

Thus $hr = h'r'$. Moreover, since the vertical angles $\angle HPS$ and $\angle H'PS'$ are equal, it follows that the right triangles $\triangle SHP$ and $\triangle S'H'P$ are similar, and thus $s/h = s'/h'$. We conclude that $rs = (hr)(s/h) = (h'r')(s'/h') = r's'$. This equality also follows from trigonometry since the area of $\triangle RPS$ is equal to $(1/2)rs \sin \angle RPS$.

Similarly, since $\triangle SPT$ and $\triangle S'PT'$ have the same area, we conclude that $st = s't'$. Combining these yields $r/t = (rs)/(st) = (r's')/(s't') = r'/t'$. Now notice that the vertical angles $\angle RPT$ and $\angle R'PT'$ are equal, so it follows that triangles $\triangle RPT$ and $\triangle R'PT'$ are similar. Thus $\angle PTR = \angle PT'R'$ and, since TT' is a transversal cutting lines a and b, we conclude that these two lines are parallel.

SECOND SOLUTION. Imagine rotating the lower part of the figure 180° about point P so that the new position of R' is some point R'' along PR. Similarly, the new locations S'' and T'' of S' and T' lie on PS and PT. Now $\triangle R''PS''$ is a rotated version of $\triangle R'PS'$, and so it has area equal to that of $\triangle RPS$. It thus cannot happen that $\triangle RPS$ lies inside $\triangle R''PS''$ with space left over, nor can it be that $\triangle R''PS''$ is contained within $\triangle RPS$ with space left over. It follows that either $R = R''$ and $S = S''$, or else line segment $\overline{RS}$ crosses line segment $\overline{R''S''}$ at some point between R and S. In either case, we see that $\overline{RS}$ and $\overline{R''S''}$ have a common point other than S. Similarly, segments $\overline{ST}$ and $\overline{S''T''}$ have a common point other than S, and it follows that lines RT and $R''T''$ have at least two distinct points in common. Thus $RT = R''T''$, and since $R''T''$ was obtained from $R'T'$ by a 180° rotation, it follows that RT and $R'T'$ are parallel, as desired.

3. Let S be a set of positive integers containing 1, 2, 3 and 4. Suppose that for every subset of S consisting of four distinct integers, the sum of that subset is also a member of S. Prove that 1000 is a member of S.

SOLUTION. Since $1 + 2 + 3 = 6$, we see that given any member a of the set S, if $a > 3$, then $a + 6$ is also a member of S. Similarly, $1 + 2 + 4 = 7$, and we conclude that $a + 7$ is in S if $a \in S$ and $a > 4$. Also, $1 + 3 + 4 = 8$ and $2 + 3 + 4 = 9$, and thus $a + 8$ and $a + 9$ lie in S for any member a of S with $a > 4$. In particular $1 + 2 + 3 + 4 = 10$ is in S, and thus $16,\ 17,\ 18$ and 19 are in S. It follows that $16 + 6,\ 16 + 7,\ 16 + 8,\ 16 + 9,\ 17 + 9,\ 18 + 9$ and $19 + 9$ all lie in S, and thus S contains the consecutive numbers $22,\ 23,\ \ldots,\ 27$.

We claim now that S contains all positive integers $n \geq 22$. If this is false, suppose that n is the smallest missing integer with $n \geq 22$. Then $n \geq 28$ and $n - 6 \geq 22$. Since $n - 6 < n$, we see that $n - 6$ is not a missing number. Thus $n - 6$ is in the set and it follows that $(n - 6) + 6 = n$ is also in S. This contradicts the choice of n as a missing number and proves that there are no missing numbers in S which are ≥ 22. In particular, $1000 \in S$.

4. Find all positive integers n such that $n^2 + 3n + 1$ is a multiple of $3n + 10$.

SOLUTION. Note that $n(3n + 10) = 3n^2 + 10n$ is a multiple of $3n + 10$. Also $3(n^2 + 3n + 1) = 3n^2 + 9n + 3$ is a multiple of $3n + 10$ by assumption. Thus

$$(3n^2 + 10n) - (3n^2 + 9n + 3) = n - 3$$

is a multiple of $3n + 10$. But $n - 3$ is an integer (possibly negative) which is clearly smaller in absolute value than $3n + 10$. Thus the divisibility implies that $n - 3 = 0$ and $n = 3$ is the only possible solution.

To check that $n = 3$ is really a solution to the problem, we observe that when $n = 3$, we have $3n + 10 = 19$ and $n^2 + 3n + 1 = 19$. Thus $n^2 + 3n + 1$ is a multiple of $3n + 10$ in this case.

5. Find all real numbers x such that

$$\sqrt[3]{x + 4} - \sqrt[3]{x} = 1$$

SOLUTION. Let us write $t = \sqrt[3]{x}$ so that our equation becomes $\sqrt[3]{x + 4} = 1 + t$. Cubing both sides, we get

$$x + 4 = 1 + 3t + 3t^2 + t^3$$

and, since $t^3 = x$, this equation simplifies to $4 = 1 + 3t + 3t^2$. Thus $t^2 + t - 1 = 0$ and the quadratic formula yields

$$t = \frac{-1 \pm \sqrt{1 + 4}}{2} = \frac{-1 \pm \sqrt{5}}{2}.$$

It follows that there are only two possible solutions to the original problem, namely $x = t^3 = -2 \pm \sqrt{5}$. A calculator check confims that each of these is really a solution.