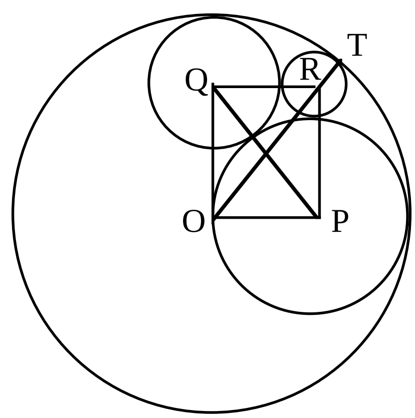
WISCONSIN MATHEMATICS SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET V (1995-96)

1. Let n be a positive integer and let x, y and z be real numbers with x and y both ≥ 1. Suppose $x^n + y^n = z^n + 1$ and $x + y = z + 1$. Show that at least one of x, y or n is equal to 1.

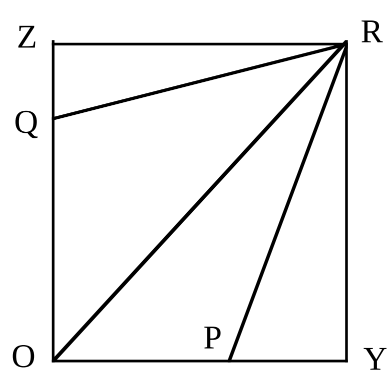
SOLUTION. Assume that $n \geq 2$ and let $y = 1+w$ so that $w \geq 0$. Then $z+1 = x+y = x+1+w$ implies that $z = x + w$, and the nth power equation becomes $x^n + (1 + w)^n = (x + w)^n + 1$. If we expand $(1 + w)^n$ and $(x + w)^n$ by the binomial theorem, then we see that the terms x^n, w^n and 1 appear on both sides of the nth power equation and hence they cancel. Consequently, the sum of the "inner terms" of $(1 + w)^n$ is equal to the corresponding sum for $(x + w)^n$. For example, when $n = 4$ we get $4w + 6w^2 + 4w^3 = 4x^3w + 6x^2w^2 + 4xw^3$ and observe that $4w \leq 4x^3w$, $6w^2 \leq 6x^2w^2$ and $4w^3 \leq 4xw^3$ since $1 \leq x$ and $0 \leq w$. Thus the right-hand side is strictly larger than the left-hand side unless $x = 1$ or $w = 0$. Of course, $w = 0$ implies that $y = 1 + w = 1$. This argument works for all $n \geq 2$. When $n = 1$ there are no "inner terms" to compare.

2. In the figure we see five circles. The largest of these (labeled A) has radius r. Each of the two circles labeled B has radius $r/2$, and they are tangent to each other and to A. Circle C is tangent to A and to both circles B and, as we saw in Problem Set III, C has radius $r/3$. Finally, circle D is tangent to A, C and to one of the circles B. Compute its radius.



SOLUTION. To avoid fractions, let us assume that $r = 6$. Let O be the center of circle A, P be the center of the right-hand circle B, Q be the center of C and R the center of D. Since B has radius 3 and C has radius 2, it follows that $OP = 3$, $PQ = 3 + 2 = 5$ and hence $OQ = 4$ since $\angle QOP = 90^\circ$. Moreover if circle D has radius x, then $PR = 3 + x$ and $QR = 2 + x$. If T is the point of tangency of circles A and D, then $\overline{OT}$ is a radius for A and $\overline{RT}$ is a radius for D. Thus $OR = OT - RT = 6 - x$.

Let us redraw the quadrilateral $OPRQ$ in a rather skewed manner and drop perpendiculars from R to extended lines $\overline{OP}$ and $\overline{OQ}$ to obtain the second figure. Set $PY = y$ and $QZ = z$, so that $ZR = OY = 3+y$ and $YR = OZ = 4+z$. (We will allow y to be negative if point Y is to the left of P. Also z will be negative if point Z is below Q.) By applying the Pythagorean theorem to right triangles PYR, QZR and OYR in turn, we obtain the equations $y^2+(4+z)^2 = (3+x)^2$, $(3+y)^2+z^2 = (2+x)^2$ and $(3+y)^2+(4+z)^2 = (6-x)^2$. By subtracting the first equation from the third, we see that $y = 3 - 3x$, and by subtracting the second equation from the third, we get $z = 2 - 2x$. Finally, substituting these expressions for y and z into any of the three equations yields the quadratic $0 = x^2-4x+3 = (x-1)(x-3)$. Since $x = 3$ is clearly too large, we must have $x = 1$. Thus $y = z = 0$ and $OPRQ$ turns out to be a rectangle. In general, the radius of circle D is (1/6)th that of circle A.



3. I have a multidigit number n whose units digit is a 1. My friend copied the number, but accidently transposed two adjacent digits to obtain the number m different from n. Find the largest possible integer which can be a divisor of both n and m.

SOLUTION. Consider for example the number $n = 729081$. If the two transposed digits are (say) the 9 and the 0, we get $m = 720981$. A quick check shows that each of n and m is divisible by 81.

Now let n and m be any numbers obtained as above. We will show that no number exceeding 81 can ever divide both of them. Suppose a and b are the two transposed digits, where a is the 10^{k+1}-digit and b is the 10^k-digit. Then we can write $n = L + a{\cdot}10^{k+1} + b{\cdot}10^k + R$, where L is the contribution to n of all the digits to the left of a and where R is the contribution of all digits to the right of b. (In the example, where $n = 729081$, $a = 9$ and $b = 0$, we see that $L = 720000$, $R = 81$ and $k = 2$.) It follows that $m = L + b{\cdot}10^{k+1} + a{\cdot}10^k + R$.

Now suppose that d is a positive divisor of both n and m. Then the difference $n - m$ must also be a multiple of d. Note that $n - m = (a{\cdot}10^{k+1} + b{\cdot}10^k) - (b{\cdot}10^{k+1} + a{\cdot}10^k) = 9a{\cdot}10^k - 9b{\cdot}10^k = 9{\cdot}10^k(a - b)$. Since d divides n and the units digit of n is a 1, we see that d must be odd and that it is not a multiple of 5. But d divides $9{\cdot}10^k(a - b)$, so it follows that d divides $9(a - b)$, and therefore $d \le 9|a - b|$. The largest possibility for $|a - b|$ occurs when $a = 9$ and $b = 0$ (or vice versa) and, in that case, $9|a - b| = 81$. Thus $d \le 81$, as claimed.

4. Find all positive integers n such that each of the three numbers $6n^2 + 5$, $2n^2 + 3$ and $n^2 + 1$ is prime.

SOLUTION. Let's experiment a bit. If $n = 1$, the three numbers are 11, 5 and 2 (three primes). When $n = 2$, the numbers are 29, 11 and 5 (again three primes). For $n = 3$, we get the set $\{59, 21, 10\}$; for $n = 4$, we get $\{101, 35, 17\}$ and when $n = 5$, the resulting set is $\{155, 53, 26\}$.

We observe that in each case, one of the three numbers is a multiple of 5 and we ask if this pattern continues. Now, the number n must have one of the forms $5k$, $5k \pm 1$ or $5k \pm 2$. If $n = 5k$, then $6n^2 + 5 = 150k^2 + 5 = 5(30k^2 + 1)$, a multiple of 5. If $n = 5k \pm 1$, then $2n^2 + 3 = 2(5k \pm 1)^2 + 3 = 50k^2 \pm 20k + 5 = 5(10k^2 \pm 4k + 1)$, which is another multiple of 5. Finally, when $n = 5k \pm 2$, we compute $n^2 + 1 = 5(5k^2 \pm 4k + 1)$, also a multiple of 5.

In all case, therefore, it is true that one of the three numbers $6n^2 + 5$, $2n^2 + 3$ and $n^2 + 1$ is a multiple of 5. If this number is prime, then certainly it must equal 5. Now $6n^2 + 5$ can never be 5 when n is a positive integer. We see that $2n^2 + 3 = 5$ only when $n = 1$ and that $n^2 + 1 = 5$ only when $n = 2$. The only possibilities, therefore, are $n = 1$ and $n = 2$, and we have seen that we actually get three primes for each of these.

5. Recall from a previous problem set that a cut-and-flip operation on a deck of 52 cards is the process of removing the top k cards from the deck, inverting this stack of k cards, and then replacing the inverted stack on the top of the deck. If all the cards in the deck are face down, show that it is possible to find a sequence of cut-and-flip operations that interchanges the 19th and 20th cards (counting from the top), leaves all the other cards in their original positions, and leaves all the cards face down.

SOLUTION. If we let $cf(k)$ denote the cut-and-flip operation which starts by removing the top k cards, then the sequence of operations $cf(20)$, $cf(1)$, $cf(2)$, $cf(1)$, $cf(20)$ will do the required interchange.