WISCONSIN MATHEMATICS SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET IV (1995-96)

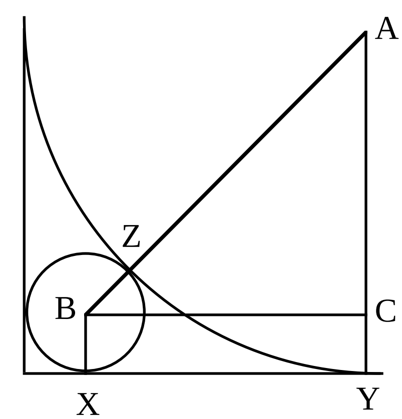
1. Consider the following sequence of numbers: 4, 12, 32, 80, 192, . . ., where the formula for the nth number is $(n+1)\,2^n$. Find a formula for the average of the first n numbers in the sequence.

SOLUTION. The answer is 2^{n+1}. It is easy to check this for the first several values of n. To prove the formula in general, we can use mathematical induction.

Assuming that we have already established that the average of the first $n-1$ numbers is 2^n, we want to deduce that the average of the first n numbers is 2^{n+1}. If we write S to denote the sum of the first n numbers in the list, then the sum of the first $n-1$ is $S-(n+1)\,2^n$, so the average of the first $n-1$ is $(S-(n+1)\,2^n)/(n-1)$. Since we know (by assumption) that this average is 2^n, we have $[S-(n+1)\,2^n]/(n-1)=2^n$ and thus $S=(n-1)\,2^n+(n+1)\,2^n=(2n)\,2^n=n\,2^{n+1}$. The average of the first n numbers is therefore equal to $S/n=2^{n+1}$, as required.

2. In the figure, a circle of radius 1 is tangent to two perpendicular lines. A second smaller circle is drawn tangent to the first circle and to the two lines, and a third circle is tangent to the second and the two lines. Imagine continuing this process until a total of ten circles have been drawn. Find the radius of the tenth circle.

SOLUTION. Suppose that two consecutive circles obtained in this process are centered at A and B, respectively. Assume that the larger circle has radius r and that the smaller one has radius s. If we drop perpendiculars $\overline{AY}$ and $\overline{BX}$ to the base line $\overline{XY}$, then $\overline{AY}$ is a radius of the larger circle, so $AY=r$, and similarly $BX=s$. Furthermore, if we draw $\overline{BC}$ perpendicular to $\overline{AY}$, as indicated, then $CY=BX=s$, so $AC=AY-CY=r-s$. Note also that the line $\overline{AB}$ joining the two centers passes through the point of tangency Z and hence $AB=AZ+BZ=r+s$. Finally, since $\angle ABC$ is clearly equal to 45°, we have $BC=AC$ and therefore



$$(r+s)^2=AB^2=AC^2+BC^2=2(r-s)^2.$$

Consequently, $r+s=\sqrt{2}(r-s)$ and we deduce that $s=r(\sqrt{2}-1)/(\sqrt{2}+1)$. In other words, at each step of this process, the radius gets smaller by a factor of $(\sqrt{2}-1)/(\sqrt{2}+1)\approx .1716$. Since we start with a circle of radius 1 and since the 10th circle is obtained in 9 steps, it follows that the radius of the 10th circle is equal to $[(\sqrt{2}-1)/(\sqrt{2}+1)]^9\approx 1.288\times 10^{-7}$.

3. Given a positive integer $B\neq 6$, show that it is possible to find a prime number p so that $9p^2+Bp+1$ is not a perfect square.

SOLUTION. We will show that if p is a prime number and if $9p^2+Bp+1$ is a perfect square, then $p\leq (B+8)/7$. In particular, if p is any prime larger than $(B+8)/7$, then $9p^2+Bp+1$ cannot be a square.

Suppose now that $9p^2+Bp+1=m^2$, where m is a nonnegative integer. Note that $m^2 > 9p^2$, so $m > 3p$. Furthermore, $m \neq 3p+1$ since $B \neq 6$, so $m > 3p+1$. Next we see that $9p^2+Bp = m^2-1 = (m-1)(m+1)$, so p divides the product $(m-1)(m+1)$. Hence, since p is prime, p must divide $m-1$ or $m+1$. This shows that $m = kp \pm 1$ for some integer k and, since $m > 3p+1$, we conclude that $m \geq 4p-1$. Thus

$$9p^2+Bp+1 = m^2 \geq (4p-1)^2 = 16p^2-8p+1,$$

so $B \geq 7p-8$ and $(B+8)/7 \geq p$.

4. Recall that if n is a positive integer, then $n!$ is the product of all the integers from 1 to n, inclusive. Show that $(2^{10})! > 2^{2^{13}}$.

SOLUTION. Note that $2^{10} = 1024$. Group the numbers $1, 2, 3, \ldots, 1023$ as follows. The first group consists of just the number 1; the second group consists of 2 and 3; the third group consists of 4, 5, 6 and 7; and in general, the kth group consists of all the numbers from 2^{k-1} to 2^k-1, inclusive. Observe that there are ten groups in total and that the kth group contains exactly 2^{k-1} numbers, each of which is $\geq 2^{k-1}$. Writing P_k to denote the product of the numbers in the kth group, we see that

$$P_k \geq (2^{k-1})^{2^{k-1}} = 2^{(k-1)\,2^{k-1}}.$$

Finally, observe that

$$1023! = P_1 P_2 P_3 \cdots P_{10} \geq 2^{0\cdot 1}\, 2^{1\cdot 2}\, 2^{2\cdot 4}\, 2^{3\cdot 8}\, 2^{4\cdot 16} \cdots 2^{9\cdot 512} = 2^m$$

where $m = 1\cdot 2 + 2\cdot 4 + 3\cdot 8 + 4\cdot 16 + \cdots + 9\cdot 512$. Using a calculator, or some facts from Problem 1, we compute that $m = 8194 = 2^{13}+2$. Thus $(2^{10})! > 1023! \geq 2^m > 2^{2^{13}}$, as required.

5. For each integer $n \geq 0$, we have a rule which gives a new integer, denoted by n^*. Suppose that

$$\frac{(n+1)^* + (n-1)^*}{2} = n^* + 1$$

for all $n \geq 1$. If $0^* = 0$ and $100^* = 20{,}000$, find 200^*.

SOLUTION. Since $(n+1)^* = 2[n^*+1] - (n-1)^*$, we can determine $(n+1)^*$ from n^* and $(n-1)^*$. In particular, if we set $1^* = 1+a$, then using $0^* = 0$, we obtain $0^* = 0+0a$, $1^* = 1+1a$, $2^* = 4+2a$, $3^* = 9+3a$, and it appears that $n^* = n^2+na$ for all $n \geq 0$. Indeed, we can verify this formula for n^* by mathematical induction. Assuming that we have already established the formula for $n-1$ and n, we need to show that it also holds for $n+1$. But this follows, since

$$\begin{aligned}(n+1)^* &= 2[n^*+1] - (n-1)^* \\ &= 2[n^2+na+1] - [(n-1)^2 + (n-1)a] \\ &= [2n^2+2-n^2+2n-1] + [2n-n+1]a \\ &= (n+1)^2 + (n+1)a.\end{aligned}$$

Finally, by hypothesis, $20{,}000 = 100^* = (100)^2 + (100)a$, so $a = 100$ and

$$200^* = (200)^2 + (200)a = 40{,}000 + 20{,}000 = 60{,}000.$$