WISCONSIN MATHEMATICS SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET III (1995-96)

1. A deck of 52 cards has some cards face up and some face down. We are allowed to rearrange this deck using a sequence of cut-and-flip operations. Specifically, a cut-and-flip operation is achieved by removing a stack of cards from the top of the deck, turning this stack over, and then returning the inverted stack to the top of the deck. Note that turning over the entire deck or just turning over the top card are examples of cut and flip operations. Prove that it is possible to get all the cards face down using no more than 52 such operations.

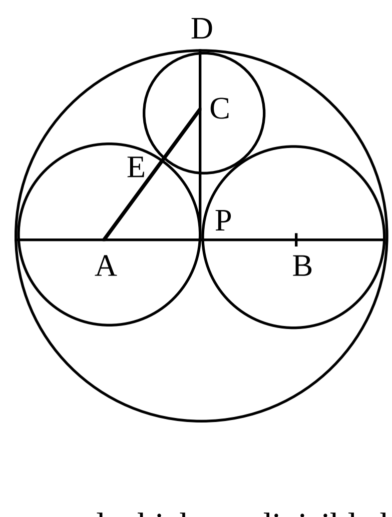
SOLUTION. We will show how to get all the cards facing the same way using no more than 51 cut-and-flip operations. If the result of this is that the cards are all face up, then one more operation (flipping the entire deck) will get them all face down, as required.

We can assume that the cards do not all face the same way at the start. Working down from the top of the deck, locate the first card that faces opposite from the top card, and suppose this is the kth card, where $1 < k \le 52$. With one cut-and-flip operation, invert the top $k-1$ cards, so that the top k cards all face the same way. It may be that all the cards now face the same way, which is our goal. Otherwise, find the first card that faces opposite from the top k cards and flip all the cards above it. Continuing like this, all the cards will eventually face the same way.

How many cut-and-flip operations were done? There was one for each wrong-facing card that we found below the top card, and so there were no more than 51 such operations. As we indicated above, this completes the proof.

2. Shown here are four circles, each tangent to the other three. The largest of these has radius 2 and each of the two medium-sized circles has radius 1. Find the radius of the smallest circle and justify your answer.

SOLUTION. Let A, B, P and C be the centers of the four circles, as shown, and observe that P is also the point of tangency of the two medium-sized circles. Also, the common tangent line to these two circles must go through C and D, where the latter is the point of tangency of the small and large circles. Furthermore, the line $\overline{AC}$ goes through the point E where the circles centered at A and C are tangent.



Let r denote the radius of the small circle and consider the right triangle $\triangle CAP$. We have $AP = 1$ and $AC = AE + EC = 1 + r$. Also, $PC = PD - DC = 2 - r$, and the Pythagorean theorem yields $(1+r)^2 = 1^2 + (2-r)^2$. Thus $1 + 2r + r^2 = 5 - 4r + r^2$, so we get $6r = 4$ and $r = 2/3$.

3. Find all positive integers which have precisely 36 positive divisors and which are divisible by precisely nine of the numbers from 1 to 10.

SOLUTION. Write $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ where the p_i are distinct prime numbers and the a_i are positive integers. Then any positive divisor m of n is of the form $m = p_1^{b_1} p_2^{b_2} \cdots p_k^{b_k}$ with $0 \le b_i \le a_i$. Thus there are $a_1 + 1$ choices for b_1, $a_2 + 1$ choices for b_2, and so on, up to $a_k + 1$

choices for b_k. We conclude that n has precisely $(a_1+1)(a_2+1)\cdots(a_k+1)$ positive divisors. In particular, if n has precisely 36 divisors, then $(a_1+1)(a_2+1)\cdots(a_k+1) = 36 = 2\cdot 2\cdot 3\cdot 3$, so n has at most 4 distinct prime factors.

If 7 does not divide n, then n is divisible by the remaining numbers ≤ 10. In particular, n is divisible by 2^3, 3^2 and 5, so $2^3 3^2 5^1$ divides n. Using $p_1 = 2$, $p_2 = 3$ and $p_3 = 5$ in the previous formula for n, we have $a_1 \geq 3$, $a_2 \geq 2$ and $a_3 \geq 1$. It then follows from the factorization of 36 that $a_1 = 3$, $a_2 = 2$, $a_3 = 2$, or that $a_1 = 5$, $a_2 = 2$, $a_3 = 1$. Furthermore, no other primes occur here, so $n = 2^3 3^2 5^2 = 1800$ or $n = 2^5 3^2 5^1 = 1440$.

Now suppose that 7 divides n and again use the information about the divisors of n between 1 and 10. For example, if 4 does not divide n, then neither does 8, and n can be divisible by at most $10-2=8$ numbers ≤ 10, a contradiction. Thus 4 must divide n and similarly we see that 3 divides n and 5 divides n. Since n is also a multiple of 7, we see that $2^2 3^1 5^1 7^1$ divides n, and since n is divisible by at most 4 primes, we see that $p_1 = 2$, $p_2 = 3$, $p_3 = 5$ and $p_4 = 7$ are all of its prime divisors. Also $a_1 \geq 2$, so $a_1 + 1 = 3$ and $a_1 = 2$. This implies that 8 does not divide n, and therefore that 9 must divide n. Thus $a_2 \geq 2$, so $a_2 + 1 = 3$, $a_3 + 1 = 2$, $a_4 + 1 = 2$ and $n = 2^2 3^2 5^1 7^1 = 1260$.

4. (New Year's Problem) Find all positive integers n for which $\sqrt{n+\sqrt{1996}}$ exceeds $\sqrt{n-1}$ by an integer.

SOLUTION. Write $\sqrt{n+\sqrt{1996}} = m + \sqrt{n-1}$, where m is a positive integer. Squaring both sides, we get $n + \sqrt{1996} = m^2 + 2m\sqrt{n-1} + n - 1$, and hence $\sqrt{1996} = m^2 - 1 + 2m\sqrt{n-1}$. In particular, this shows that $m^2 \leq 1 + \sqrt{1996} \approx 45.7$, and we deduce that $m < 7$.

There are just six possibilities for m since m is an integer and $1 \leq m \leq 6$. For each possible value for m, we have $\sqrt{n-1} = (1+\sqrt{1996}-m^2)/2m$, and thus $n = 1+(1+\sqrt{1996}-m^2)^2/(2m)^2$. If we plug in the six possible values for m and use a calculator, we get the following approximate values for n: 500.00 when $m = 1$; 109.56 when $m = 2$; 38.37 when $m = 3$; 14.76 when $m = 4$; 5.28 when $m = 5$; and 1.65 when $m = 6$. The only possibility, therefore, is that $n = 500$ and $m = 1$. To see that this actually is a solution, we must show that $500 = (\sqrt{1996}/2)^2 + 1$, and this follows since $(\sqrt{1996}/2)^2 = 1996/4 = 499$. Thus $n = 500$ is the only solution to the problem.

5. Consider the equation $(x^2 + 10x + a)^2 = b$. (i) If $a = 21$, find a number b so that there are exactly three real values of x which satisfy the equation. (ii) If $a \geq 25$, show that there are no numbers b for which the equation has exactly three solutions.

SOLUTION. Fix the number a and suppose that the equation $(x^2 + 10x + a)^2 = b$ has precisely three solutions. Then certainly $b > 0$, so say $b = c^2$ for some positive real number c. Since $(x^2+10x+a)^2 = c^2$, the equation reduces to the two quadratic equations $x^2 + 10x + (a+c) = 0$ and $x^2 + 10x + (a-c) = 0$. Thus one of these quadratics must have two distinct solutions and the other must have a double root. By the quadratic formula, the roots of the two equations are $-5\pm\sqrt{25-a-c}$ and $-5\pm\sqrt{25-a+c}$, respectively. In particular, we must have $25-a-c \geq 0$ and $25-a+c \geq 0$. In fact, one of these must be 0 to yield the double root and the other is strictly bigger than 0 to yield two distinct roots. Since $c > 0$, we therefore have $25-a-c = 0$ and $25-a+c > 0$, and by adding the latter two relations, we obtain $2(25-a) > 0$. In other words, $a < 25$. Note that, when $a = 21$, we have $c = 25 - a = 4$, so $b = c^2 = 16$.