WISCONSIN MATHEMATICS SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET II (1995-96)

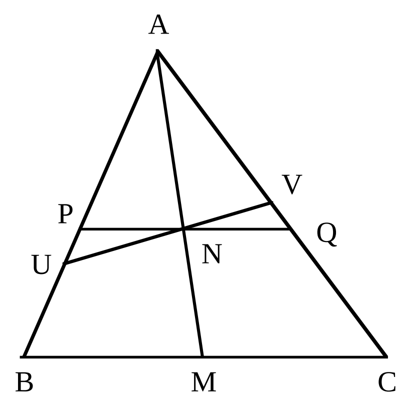
1. Sam and Pat keep giving money to each other. They start with a total of $64,000 between them, and each gift is exactly enough to double the assets of the recipient. They find that eventually Sam has possession of all the money. If each player started with a whole number of dollars, show that each of these numbers must have been a multiple of 125.

SOLUTION. At the end of the game, Pat has $0 while Sam has $64,000, and we observe that each of 0 and 64,000 is a multiple of 125. To prove that at the beginning it was also true that each player held a multiple of 125 dollars, it suffices to show that if this condition holds at some stage of the game, then it must also have held at the previous stage.

Suppose that at the previous stage, one player had a dollars and the other had b dollars, where a and b are integers and $a \leq b$. Then a dollars changes hands and at the next stage, the players assets are $2a$ dollars and $b - a$ dollars. We are assuming that each of these is a multiple of 125. In particular, 125 divides $2a$ and, since 125 is odd, we see that 125 divides a. Furthermore, 125 divides a and $b - a$, so 125 also divides $a + (b - a) = b$. This completes the proof.

2. Let U and V be points on sides $\overline{AB}$ and $\overline{AC}$ of $\triangle ABC$ as shown. Suppose that the median $\overline{AM}$ bisects $\overline{UV}$. Prove that $\overline{UV}$ is parallel to $\overline{BC}$.

SOLUTION. Suppose that $\overline{UV}$ is not parallel to $\overline{BC}$. Let $\overline{UV}$ and $\overline{AM}$ meet at N, and draw line segment $\overline{PQ}$ through N parallel to $\overline{BC}$. Here P lies on $\overline{AB}$ and Q lies on $\overline{AC}$, as shown. Since $\overline{PQ} \parallel \overline{BC}$, we know that $\triangle APN \sim \triangle ABM$ and also that $\triangle ANQ \sim \triangle AMC$. From these similarities we deduce that $PN/BM = AN/AM = QN/CM$. Thus $PN/QN = BM/CM = 1$, so $PN = QN$ and N is the midpoint of $\overline{PQ}$. We are given that $UN = VN$ and we know that $\angle UNP = \angle VNQ$ since these are vertical angles. Thus SAS implies that $\triangle UNP \cong \triangle VNQ$, and hence $\angle UPN = \angle VQN$. From the latter we deduce that $\overline{UP}$ and $\overline{VQ}$ are parallel, a ridiculous statement since lines $\overline{UP}$ and $\overline{VQ}$ meet at A. Since this contradiction resulted from our initial assumption that $\overline{UV}$ is not parallel to $\overline{BC}$, it follows therefore that these lines must actually be parallel.



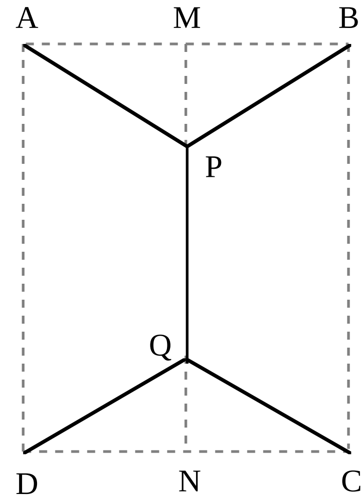
3. Find all pairs of positive integers x and y such that $x^2 + y$ exceeds $x + y^2$ by precisely 10.

SOLUTION. We want to solve the equation $(x^2 + y) - (x + y^2) = 10$ for positive integers x and y. Rewriting the left side of this equation as $(x^2 - y^2) - (x - y)$, we see that it factors as $(x - y)(x + y - 1)$. We must therefore solve the equation $(x - y)(x + y - 1) = 10$.

Since $x \geq 1$ and $y \geq 1$, we see that $x + y - 1$ is positive and thus $x - y = 10/(x + y - 1)$ must also be positive. Also, $x - y < x + y - 1$, and thus the equation $(x - y)(x + y - 1) = 10$ represents a factorization of 10 into two positive integers, of which the first is smaller than the second. There are just two possibilities. Either $x - y = 1$ and $x + y - 1 = 10$ or else $x - y = 2$ and $x + y - 1 = 5$. In the first case, we get $x = 6,\ y = 5$ and in the second, we see that $x = 4,\ y = 2$.

4. Four towns are located at the vertices of a 4 mile by 6 mile rectangle. By using three sides of the rectangle, a road network of total length 14 miles can be constructed which connects all four towns. Is there a shorter road network which connects the towns? Specifically, is there a suitable network of length less than 13 miles?

SOLUTION. Consider the road network made of five line segments, as illustrated in the figure. Towns A, B, C and D form the vertices of a rectangle with $AB = 4$ and $AD = 6$. We have created road junctions P and Q so that $\triangle APB$ and $\triangle CQD$ are isosceles with $\angle APB = 120^\circ = \angle CQD$. In this situation, $\overline{PQ}$ is parallel to $\overline{AD}$ and meets $\overline{AB}$ and $\overline{CD}$ at M and N, as shown.



In the right triangle $\triangle AMP$, we see that $\angle APM = 60^\circ$, and it follows that $AP = 2\,MP$. From this, we see that $AM = \sqrt{3}\,MP$. But $AM = \frac{1}{2}\,AB = 2$, so we conclude that $MP = 2/\sqrt{3}$ and $AP = 4/\sqrt{3}$. The total length of the four diagonal lines in the figure is thus $16/\sqrt{3}$. Also $NQ = MP = 2/\sqrt{3}$ and $MN = 6$. It follows that $PQ = 6 - 4/\sqrt{3}$ and the total length of our network is

$$\begin{aligned} 16/\sqrt{3} + \left(6 - 4/\sqrt{3}\right) &= 6 + 12/\sqrt{3} \\ &= 6 + 4\sqrt{3} \approx 12.928 \text{ miles.} \end{aligned}$$

This is less than 13 miles, as required.

5. Let f be a rule which assigns a nonnegative integer called $f(x)$ to each positive integer x. We say that f is a "product-sum rule" if $f(xy) = f(x) + f(y)$ for all positive integers x and y. (a) Find a product-sum rule f such that $f(x)$ is never zero when $x > 1$. (b) If f is a product-sum rule, show that there exist distinct positive integers x and y such that $f(x) = f(y)$.

SOLUTION. (a) An easy way to create a product-sum rule is to let $f(n)$ be the total number of prime factors of n, counting repeats. For example, since $72 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3$, we set $f(72) = 5$. Similarly, $f(300) = 5$ since $300 = 2 \cdot 2 \cdot 3 \cdot 5 \cdot 5$. We set $f(1) = 0$.

(b) Now let f be any product-sum rule and observe that $f(1) = f(1 \cdot 1) = f(1) + f(1)$, so $f(1) = 0$. If also $f(3) = 0$, then we can take $x = 1$ and $y = 3$, and thus $f(x) = 0 = f(y)$, as required. We can therefore assume that $f(3) \neq 0$.

In this case, let $x = 2^{f(3)}$ and $y = 3^{f(2)}$. Since f is a product-sum rule and x is a product of $f(3)$ copies of 2, we see that $f(x)$ is the sum of $f(3)$ copies of $f(2)$. In other words, $f(x) = f(3) \cdot f(2)$, and similarly $f(y) = f(2) \cdot f(3)$. Thus $f(x) = f(y)$ and what remains is to show that $x \neq y$. But $x = 2^{f(3)}$ and $f(3) \neq 0$, so we see that x is an even integer. On the other hand, $y = 3^{f(2)}$ is odd, and thus $x \neq y$, as desired.