WISCONSIN MATHEMATICS SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET I (1995-96)

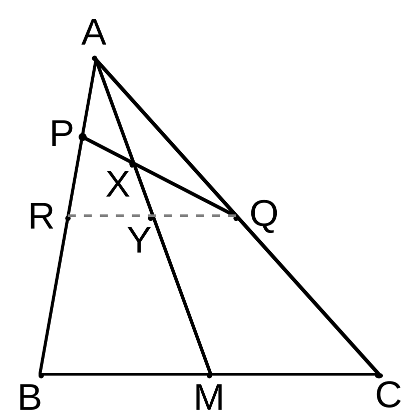
1. Ten points are chosen on a circle and all of the chords determined by these points are drawn.
(a) How many chords are there?
(b) Assume that no three of these chords intersect at a common point inside the circle. How many points inside the circle lie on two chords?

SOLUTION. Each of the chords has two of the original ten points as its endpoints, and every choice of two of these points determines a chord. The number of chords is therefore the number of ways of choosing two points from ten. This number is $(10 \cdot 9)/(2 \cdot 1) = 45$.

Now each interior intersection point has exactly two chords going through it and thus it determines four distinct endpoints. Conversely, every choice of four of the original ten points determines six chords, but these have just one interior intersection point. The total number of interior intersection points is therefore equal to the number of ways of choosing four points from ten, namely $(10 \cdot 9 \cdot 8 \cdot 7)/(4 \cdot 3 \cdot 2 \cdot 1) = 210$.

2. In $\triangle ABC$, points P and Q are chosen on sides $\overline{AB}$ and $\overline{AC}$ respectively, and segment $\overline{PQ}$ is drawn meeting median $\overline{AM}$ at X. If $AP = \frac{1}{4}AB$ and $AQ = \frac{1}{2}AC$, determine the ratio PX/PQ, and prove that your answer is correct.

SOLUTION. Draw the line segment $\overline{RQ}$ parallel to $\overline{BC}$, as indicated, and let Y be the point where $\overline{RQ}$ meets $\overline{AM}$. Since $\overline{YQ}$ is parallel to $\overline{MC}$ and passes through the midpoint Q of $\overline{AC}$, we know that $YQ = \frac{1}{2}MC$ and similarly $RQ = \frac{1}{2}BC$. Since $MC = \frac{1}{2}BC$, it follows that $YQ = \frac{1}{2}RQ$ and Y is the midpoint of $\overline{RQ}$. Since $\overline{RQ}$ is parallel to $\overline{BC}$ and Q is the midpoint of $\overline{AC}$, we can also conclude that R is the midpoint of $\overline{AB}$. But $AP = \frac{1}{4}AB = \frac{1}{2}AR$, and thus P is the midpoint of $\overline{AR}$.



We now see that $\overline{AY}$ and $\overline{QP}$ are medians of $\triangle ARQ$ and hence their intersection point X lies $2/3$ of the way from Q to P. It follows that $PX/PQ = 1/3$.

3. For which integers $n \geq 2$ is $(n-1)!$ not a multiple of n? (Recall that $k!$ is the product $1 \cdot 2 \cdot 3 \cdots (k-1) \cdot k$.)

SOLUTION. Since $3! = 6$ is not a multiple of 4, we see that 4 is one of the numbers with n not dividing $(n-1)!$. Moreover, note that none of the factors $1, 2, 3, \ldots, n-1$ is divisible by n. Thus if n is prime, then again n does not divide $(n-1)!$. We will show that these are the only possibilities.

Suppose that n is not equal to 4 or a prime. Let p be a prime divisor of n and write $n = pm$. Since n is not prime, we see that $p < n$, so both p and m occur among the factors defining $(n-1)!$. In particular, if $p \neq m$, then $n = pm$ divides $(n-1)!$. On the other hand, if $p = m$, then $n = p^2$ and thus $p > 2$ since $n \neq 4$. It follows that $2p < n$, so both p and $2p$ are distinct defining factors of $(n-1)!$. Therefore $(n-1)!$ is a multiple of $p(2p) = 2n$, and thus it is also a multiple of n.

4. For which real numbers a does the equation

$$|x-1| - |x-2| + |x-4| = a$$

have exactly three solutions?

SOLUTION. We begin by graphing the function $y = f(x) = |x-1| - |x-2| + |x-4|$. If $x \le 1$, then $x - 1 \le 0$, $x - 2 \le 0$ and $x - 4 \le 0$, so we have

$$y = -(x-1) + (x-2) - (x-4) = -x + 3.$$

If $1 \le x \le 2$, then $x - 1 \ge 0$, but $x - 2 \le 0$ and $x - 4 \le 0$, so

$$y = (x-1) + (x-2) - (x-4) = x + 1.$$

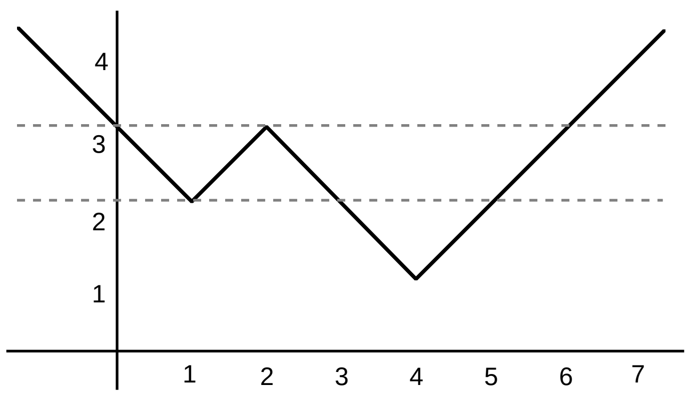
In the interval $2 \le x \le 4$ we have $x - 1 \ge 0$ and $x - 2 \ge 0$, but $x - 4 \le 0$, so

$$y = (x-1) - (x-2) - (x-4) = -x + 5.$$

Finally, for $x \ge 4$ we have

$$y = (x-1) - (x-2) + (x-4) = x - 3.$$

By piecing together the relevant parts of these four lines, we get the following graph of the function $f(x)$.



Thus, for the original equation to have exactly three solutions, we have to choose a so that the horizontal line $y = a$ touches the graph of $f(x)$ at exactly three points. As we can see, this happens only for $a = 2$ and $a = 3$.

5. Suppose that S is a set consisting of three positive integers and that the sum of every two members of S is a square. (For example, S could be $\{5, 20, 44\}$ or $\{10, 54, 90\}$.) Prove that S contains at most one odd number.

SOLUTION. Every integer has one of the four forms $4k$, $4k+1$, $4k+2$ and $4k+3$ for integers k. Square integers, on the other hand, must have one of the forms $4k$ or $4k+1$.

Now suppose that the set S contains the two odd numbers x and y. Since $x + y$ is an even square, it must have form $4k$, and therefore x and y cannot both have form $4k+1$, nor can they both have form $4k+3$. It follows that we can write $x = 4a + 1$ and $y = 4b + 3$.

We derive a contradiction by showing that there is no possibility for the third member z of S. Indeed, if z has form $4k$ or $4k+3$, then $z + y$ is not a square, and if z has form $4k+1$ or $4k+2$, then $z + x$ is a nonsquare. It follows that S cannot have as many as two odd numbers.