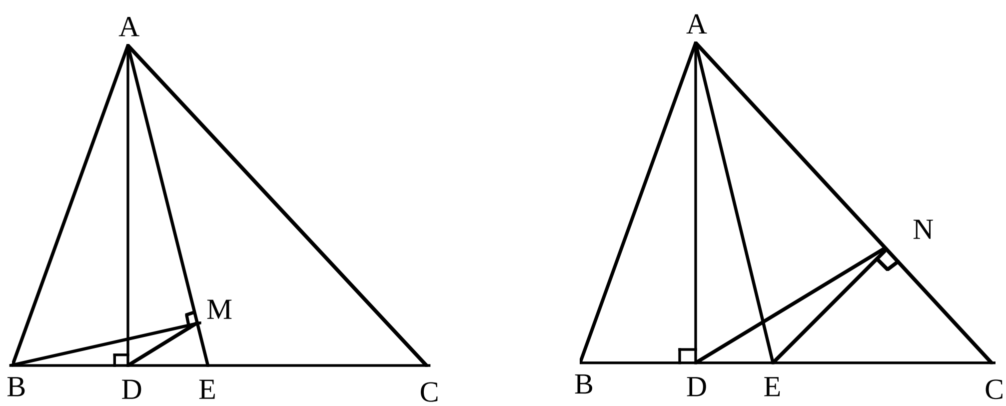
WISCONSIN MATHEMATICS SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET V (1994-95)

1. Let $\square$ be an operation (like addition or multiplication) which associates to each pair x, y of real numbers the real number $x \square y$. Suppose that, for all real x, y, z, we have (1) $x \square x = x$, (2) $x \square y = y \square x$, (3) $x \square (y \square z) = (x \square y) \square z$, and (4) if $y < z$ and $x \square y \neq x$, then $x \square y < x \square z$. In the preceding problem set, we showed that $x \square y = x$ or y for all x, y. Find infinitely many different operations $\square$ satisfying the above four conditions.

SOLUTION. For each real number r, we will define an operation depending on r. Specifically, let $x \square y = r$ if either x or y is equal to r, and $x \square y = \max\{x, y\}$ otherwise. Then $\square$ certainly satisfies (1) and (2). For (3), if one of x, y, or z is equal to r, then $x \square (y \square z) = r = (x \square y) \square z$. If none of them are equal to r, then $x \square (y \square z) = \max\{x, y, z\} = (x \square y) \square z$. Finally, we consider (4). Suppose x, y, z are given with $y < z$ and $x \square y \neq x$. Then the latter implies that $x \square y = y$ and $x \neq r$. Moreover, if $z = r$, then $x \square z = z$ and if $z \neq r$, then $x \square z = \max\{x, z\} \geq z$. Thus, in either case, $x \square z \geq z > y = x \square y$. Thus, $\square$ satisfies all four conditions. Furthermore, r is the unique real number with $r \square x = r$ for all x. Thus, each such r gives rise to a different operation $\square$ and, by varying r, we obtain infinitely many such operations.



2. In $\triangle ABC$, suppose that $AD \perp BC$ and that AE is the angle bisector of $\angle BAC$. If $BM \perp AE$ and $EN \perp AC$, prove that points D, M, and N are colinear. (Hint. Use the conclusion of Problem Set IV, Problem 2.)

SOLUTION. First, consider the quadrilateral $AMDB$. Since $\angle AMB = \angle ADB$, it follows from the conclusion of Problem Set IV, Problem 2 that $\angle BAM + \angle BDM = 180^\circ$. Furthermore, $\angle BDM + \angle EDM = 180^\circ$, so $\angle EDM = \angle BAM = (1/2)\angle BAC$, since AE is the angle bisector of $\angle BAC$. Similarly, consider quadrilateral $ADEN$. Since $\angle ADE + \angle ANE = 180^\circ$, Problem Set IV, Problem 2 implies that $\angle EDN = \angle EAN = (1/2)\angle BAC$. Thus $\angle EDN = \angle EDM$, and hence D, M, and N are colinear.

3. Which positive integers n divide

$$S(n) = 1^{1995} + 2^{1995} + \cdots + (n-1)^{1995}.$$

SOLUTION. We only use the fact that 1995 is an odd number > 1. Observe that, if a is any integer, then there exist an integer A such that

$$a^{1995} + (n-a)^{1995} = a^{1995} + nA + (-a)^{1995} = nA.$$

In other words, $a^{1995}+(n-a)^{1995}$ is a multiple of n, and this allows us to pair off certain summands in $S(n)$ and know that their sum is divisible by n. Suppose first that n is odd. Then $n-1$ is even and hence we can pair off all the summands of $S(n)$ to conclude that n divides $S(n)$. Next, suppose that n is even. Then we can pair off all but the middle summand, and therefore n divides $S(n)$ if and only if n divides $(n/2)^{1995}$. Now if $n/2$ is even, then n divides $2(n/2)$ and therefore n divides $(n/2)^{1995}$. On the other hand, if $n/2$ is odd, then the even number n cannot divide the odd number $(n/2)^{1995}$. Thus, if n is even, then n divides $S(n)$ if and only if n is divisible by 4.

4. Find all positive integers x and y which satisfy the equation $x^2+x=y^4+y^3+y^2+y$.

SOLUTION. For convenience, let us write $f(x)=x^2+x=x(x+1)$, so that f is an increasing function when x is positive. Fix the positive integer y, set $g(y)=y^4+y^3+y^2+y$, and observe that

$$g(y)-f(y^2+y/2-1/2)=(3/4)y^2+y+1/4>0$$

and that, for $y>2$, we have

$$g(y)-f(y^2+y/2)=y/2-y^2/4<0.$$

Since f is increasing, it follows that if $y>2$, and if $x>0$ with $f(x)=g(y)$, then $y^2+y/2-1/2<x<y^2+y/2$. But $y^2+y/2$ is either an integer or a half integer, so there can be no integer x satisfying the above inequality. In other words, there are no integer solutions with $y>2$. Finally, if $y=1$, then $x^2+x=g(1)=4$, and again there is no solution. On the other hand, if $y=2$, then $x^2+x=g(2)=30$ and this has the solution $x=5$. Thus $x=5$, $y=2$ is the only possibility.

5. An n-digit number α is said to be *special* if (1) α is equal to the arithmetic mean of the $n!$ numbers one obtains by rearranging the digits of α in all possible ways, and (2) the digits of α are not all equal. We know, from the preceding problem set, that the 3-digit special numbers are 370, 407, 481, 518, 592, and 629. Find the next larger special number and then show that there are infinitely many special numbers.

SOLUTION. Suppose $\alpha=a_{n-1}\cdots a_1a_0=\sum_{i=0}^{n-1}a_i10^i$. If we rearrange the digits of α in all $n!$ possible ways, then each a_i will occur precisely $(n-1)!$ times in the 1's place, in the 10's place, etc. Thus the sum of the $n!$ rearrangements is $(n-1)!s(\alpha)\sum_{i=0}^{n-1}10^i=(n-1)!s(\alpha)11\cdots11$, where $s(\alpha)=\sum_{i=0}^{n-1}a_i$ is the sum of the digits of α. Since $(n-1)!/n!=1/n$, it follows that α satisfies condition (1) if and only if $\alpha=s(\alpha)11\cdots11/n$. Note that, if α is a multiple of $11\cdots11$, then all digits of α will be equal, and α will not be special. In particular, since $n=4$ has no factor in common with 1111 and since $n=5$ has no factor in common with 11111, it follows that α will be a multiple of $11\cdots11$ in those cases and hence α will not be special. Thus the first possibility after $n=3$ occurs when $n=6$. Here $111111/3=37037$, so α must satisfy $\alpha=37037s(\alpha)/2=37037k$ for some integer k. Since α has 6 digits, $k\geq3$ and since α is not a multiple of 111111, k is not a multiple of 3. We try $k=4,5,7,8,10,\ldots$ and discover that the first possibility occurs when $k=10$. Thus the next special number, after the 3-digit ones, is 370370.

Since 370 is a special 3-digit number, we may now guess that if β is any 3-digit special number, then the $3m$-digit number $\beta\beta\cdots\beta\beta$ is also special. This is easy to verify (try it), and we conclude that there are infinitely many special numbers.