WISCONSIN MATHEMATICS SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET IV (1994-95)

1. Let $\Box$ be an operation (like addition or multiplication) which associates to each pair x, y of real numbers the real number $x \Box y$. Suppose that, for all real x, y, z, we have (1) $x \Box x = x$, (2) $x \Box y = y \Box x$, (3) $x \Box (y \Box z) = (x \Box y) \Box z$, and (4) if $y < z$ and $x \Box y \neq x$, then $x \Box y < x \Box z$. Show that $x \Box y = x$ or y for all x, y. Furthermore, find two different operations $\Box$ satisfying the above four conditions.

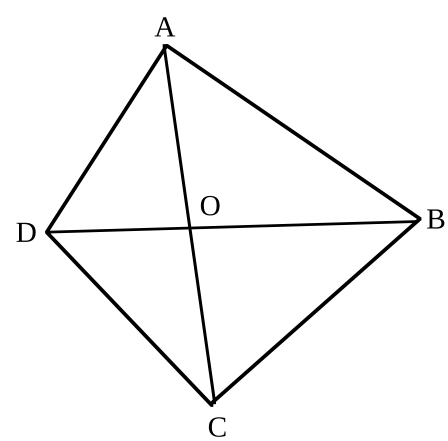
SOLUTION. Suppose x and y are real numbers with $x \Box y \neq x$ or y. By (2), we can assume that $y < x$. Since $x \Box y \neq x$, conditions (4) and (1) imply that $x \Box y < x \Box x = x$. Now note that

$$y \Box (x \Box y) = (x \Box y) \Box y = x \Box (y \Box y) = x \Box y$$

by (1), (2), and (3). Thus, since $x \Box y \neq y$, (4) yields $y \Box (x \Box y) < y \Box x$ or $x \Box y < x \Box y$, a contradiction. It follows that the supposition is false, so $x \Box y = x$ or y.

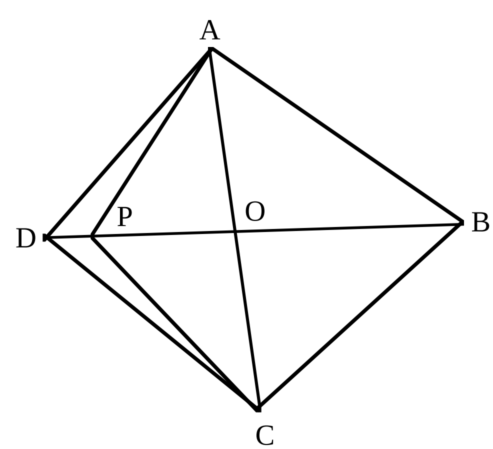
Now we offer two examples. First, consider $x \Box y = \max\{x, y\}$. Since $x \Box (y \Box z) = \max\{x, y, z\}$, it is clear that (1), (2), and (3) are satisfied. For (4), let $y < z$. If $\max\{x, y\} = x \Box y \neq x$, then $x < y$ so $x \Box y = y < z = x \Box z$. Finally, consider $x \Box y = \min\{x, y\}$, and again we know that this operation satisfies (1), (2), and (3). For (4), let $y < z$. If $\min\{x, y\} = x \Box y \neq x$, then $y < x$ so $x \Box y = y < \min\{x, z\} = x \Box z$.

2. In quadrilateral $ABCD$, show that $\angle CAD = \angle CBD$ if and only if $\angle ABC + \angle ADC = 180^\circ$.



SOLUTION. Suppose first that $\angle CAD = \angle CBD$. Since the oppositive angles at point O are equal, it follows that $\triangle OAD$ is similar to $\triangle OBC$. In particular, $OB/OA = OC/OD$ and, since $\angle AOB = \angle COD$, it follows that $\triangle AOB$ is similar to $\triangle DOC$. Hence $\angle BAC = \angle BDC$. But then the four summands which add to $\angle ABC + \angle ADC$ are equal to the four summands which add to the sum of the angles of $\triangle ABD$. Thus $\angle ABC + \angle ADC = 180^\circ$.

Conversely, suppose $\angle ABC + \angle ADC = 180^\circ$. If $\angle CAD > \angle CBD$, then we can draw the lines AP and PC so that $\angle CAP = \angle CBD$. By the above, this implies that $\angle ABC + \angle APC = 180^\circ$, so $\angle APC = \angle ADC$. But $\angle APC = \angle ADC + \angle DAP + \angle DCP$, so this is a contradiction. We obtain a similar contradiction when $\angle CAD < \angle CBD$.



An alternate argument is to show that the angle conditions are both equivalent to $ABCD$ being inscribable in a circle.

3. Solve the equation
$$x^4 + 1 = 2x(x^2 + 1).$$

SOLUTION. Observe that

$$0 = (x^4 + 1) - 2x(x^2 + 1) = x^4 - 2x^3 - 2x + 1 = (x^2 - x + 1)^2 - 3x^2$$

so $x^2 - x + 1 = \epsilon\sqrt{3}x$ where $\epsilon = \pm 1$. Thus $x^2 - (1 + \epsilon\sqrt{3})x + 1 = 0$ and, with $\delta = \pm 1$, the quadratic formula yields
$$x = \frac{1 + \epsilon\sqrt{3} + \delta\sqrt{2\epsilon\sqrt{3}}}{2}.$$

4. Suppose n is a positive integer. Find the smallest positive integer x such that 2^n divides
$$x^{1995} + 1.$$

SOLUTION. Since $x^{1995} + 1$ is even, it is clear that x is odd. Now note that

$$x^{1995} + 1 = (x + 1)(x^{1994} - x^{1993} + \cdots - x + 1)$$

and that the second factor is a sum of an odd number (1995) of odd terms and hence is odd. Thus 2^n divides $x^{1995} + 1$ if and only if it divides $x + 1$, and therefore the smallest such x is $2^n - 1$.

5. Find all 3-digit numbers m which are equal to the arithmetic mean of the six numbers one obtains by rearranging the digits of m in all possible ways.

SOLUTION. Suppose $m = abc = 100a + 10b + c$. If we rearrange the digits of m in all six possible ways, then each of a, b, and c will occur precisely twice in the 1's place, twice in the 10's place, and twice in the 100's place. Thus the sum of the six rearrangements is $2(a + b + c)(100 + 10 + 1) = 222(a + b + c)$ and hence the arithmetic mean of these six numbers is $37(a + b + c)$. By assumption,

$$100a + 10b + c = m = 37(a + b + c),$$

so $7a = 3b + 4c$ or $7(a - c) = 3(b - c)$.

Now $-9 \le b - c \le 9$ and 7 divides $b - c$, so there are just three possibilities. If $b - c = 0$, then $a = b = c$ and we obtain the numbers aaa with $a = 1, 2, \ldots, 9$. If $b - c = 7$, then $a - c = 3$, so $b = c + 7$, $a = c + 3$ and there are only three possibilities for c namely 0,1,2. These yield $m = 370, 481,$ and 592. Finally, if $b - c = -7$, then $a - c = -3$, so $b = c - 7$, $a = c - 3$. Thus $c = 7, 8, 9$ and these yield $m = 407, 518,$ and 629. Of course, one could do this all by computer, but that would be less of a challenge.