## WISCONSIN MATHEMATICS SCIENCE & ENGINEERING TALENT SEARCH

### SOLUTIONS TO PROBLEM SET III (1994-95)

**1.** Find the largest positive integer $x$, not divisible by 10, such that if the last two digits of $x^2$ are removed, then the remaining number is also a perfect square.

**SOLUTION.** Let $n$ be the two-digit number which is the last two digits of $x^2$. Then $0 \le n < 100$, and removing the last two digits of $x^2$ yields the number $(x^2 - n)/100$. By assumption, $(x^2 - n)/100 = y^2$ for some nonnegative integer $y$, and thus

$$n = x^2 - 100y^2 = (x - 10y)(x + 10y).$$

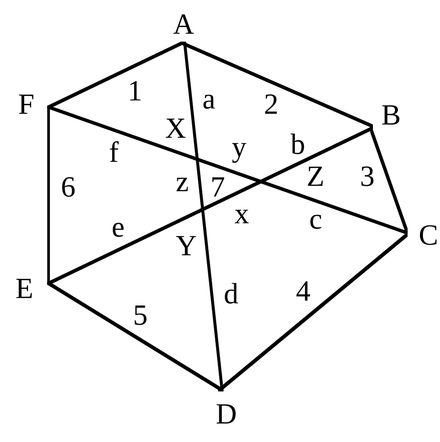
Since $x$ is not divisible by 10, it follows that $x > 10y$. Suppose $x > 40$. Then $y \ne 0$, since otherwise $x^2 = n < 100$. Thus $y \ge 1$ and $x + 10y > 50$. Now $n < 100$, so the above factorization implies that $x - 10y < 2$ and hence $x - 10y = 1$. Thus

$$100 > n = (x - 10y)(x + 10y) = 20y + 1,$$

and we conclude that $y \le 4$ and $x = 10y + 1 \le 41$. In other words, we have shown that if $x > 40$, then $x = 41$ and therefore $x = 41$ is the largest possible solution. Note that $41^2 = 1681$ and that removing the last two digits yields the perfect square 16.

**2.** Suppose that each of the three main diagonals $AD$, $BE$, and $CF$ divide the hexagon $ABCDEF$ into two regions of equal area. Prove that the three diagonals meet at a common point.

**SOLUTION.** In general, the three main diagonals divide the hexagon into seven parts, and we label these in the diagram as $1, 2, \ldots, 7$. Since each of the main diagonals divide the area of $ABCDEF$ in half, we have Area(1) + Area(2) + Area(3) = Area(2) + Area(3) + Area(4) + Area(7), and thus Area(1) = Area(4) + Area(7). In other words, the areas of the two opposite triangles $\triangle AFX$ and $\triangle CDX$ are equal. Indeed, since two triangles have equal angles at $X$, the equal areas imply that




$$af = (c + y)(d + z) \ge cd.$$

Similarly

$$bc = (e + x)(f + y) \ge ef$$
$$de = (a + z)(b + x) \ge ab.$$

But the product of the three left-hand terms above is the same as the product of the three right-hand terms, so we must have equality throughout. In particular, $x = y = z = 0$ and hence the three diagonals meet at a common point.

**3.** Find all positive integer solutions of the equation

$$x^3 - y^3 = xy + 61.$$

**SOLUTION.** Clearly $x > y \geq 1$ and $xy + 61 = x^3 - y^3 = (x - y)(x^2 + xy + y^2)$, so

$$61 = (x - y)(x^2 + y^2) + (x - y - 1)xy.$$

If $x - y \geq 3$, then $x \geq 4$ and $61 > 3(x^2 + y^2)$. Thus $x = 4$, so $y = 1$ and we note that these do not satisfy the original equation. If $x - y = 2$, then

$$61 = 2(x^2 + y^2) + xy = 2((y + 2)^2 + y^2) + (y + 2)y = 5y^2 + 10y + 8$$

and this equation also has no positive integer solution in $y$. Thus we must have $x - y = 1$, so

$$61 = x^2 + y^2 = (y + 1)^2 + y^2$$

and hence $y = 5$ and $x = 6$.

**4.** Without using a calculator or a computer, determine which of the two numbers $31^{11}$ or $17^{14}$ is larger.

SOLUTION. We have

$$31^{11} < 32^{11} = 2^{5 \cdot 11} = 2^{55} \qquad \text{and} \qquad 17^{14} > 16^{14} = 2^{4 \cdot 14} = 2^{56}.$$

Thus $17^{14} > 31^{11}$. With a calculator, we see that

$$17^{14} \approx 1.684 \times 10^{17} \qquad \text{and} \qquad 31^{11} \approx 2.541 \times 10^{16}$$

so $17^{14}/31^{11} \approx 6.627$.

**5.** Suppose that $S$ is a set of 400 consecutive integers. How many ordered pairs $(s, t)$ are there, with $s$ and $t$ in $S$, such that either $s \neq t$ and $s + t$ is divisible by 80, or $s = t$ and $s + t$ is divisible by 160.

**SOLUTION.** For each $s$ in $S$, the collection of all sums $s + t$ with $t$ in $S$ is a set of 400 consecutive integers. Thus precisely 5 of these numbers is divisible by 80. In other words, there are precisely $400 \times 5 = 2000$ ordered pairs $(s, t)$ with $s$ and $t$ in $S$ and with $s + t$ divisible by 80. Now the 400 pairs $(u, u)$ with $u$ in $S$ give rise to sums $u + u = 2u$ which are 400 "consecutive" even integers. Precisely 10 of these sums are divisible by 80 and 5 are divisible by 160. Thus there are 5 not divisible by 160 and, by deleting these from the above 2000 pairs, we obtain a collection of size $2000 - 5 = 1995$. Happy New Year!