WISCONSIN MATHEMATICS SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET II (1994-95)

1. Suppose $a_0, a_1, a_2, \ldots, a_n$ are positive real numbers satisfying $a_i a_{n-i} = 1$ for all $i = 0, 1, 2, \ldots, n$. If k is any integer, compute the sum

$$\frac{1}{1+a_0^k} + \frac{1}{1+a_1^k} + \frac{1}{1+a_2^k} + \cdots + \frac{1}{1+a_n^k}$$

SOLUTION. The answer is independent of k and just about everything else. Let

$$S = \frac{1}{1+a_0^k} + \frac{1}{1+a_1^k} + \frac{1}{1+a_2^k} + \cdots + \frac{1}{1+a_n^k}$$

and write this sum in the reverse order as

$$S = \frac{1}{1+a_n^k} + \frac{1}{1+a_{n-1}^k} + \frac{1}{1+a_{n-2}^k} + \cdots + \frac{1}{1+a_0^k}$$

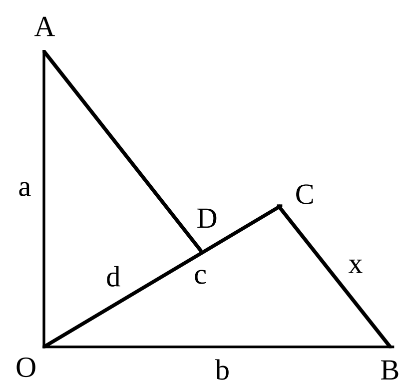
Since $a_{n-i} = 1/a_i$, we have

$$\frac{1}{1+a_i^k} + \frac{1}{1+a_{n-i}^k} = \frac{1}{1+a_i^k} + \frac{1}{1+a_i^{-k}} = \frac{1}{1+a_i^k} + \frac{a_i^k}{1+a_i^k} = 1$$

Adding the two formulas for S yields $2S = 1 + 1 + 1 + \cdots + 1 = n + 1$ and $S = (n+1)/2$.

2. Assume that $\angle AOB$ is a right angle. Find a formula for the area of $\triangle AOD$ in terms of the lengths $OA = a$, $OB = b$, $OC = c$, $OD = d$, and $BC = x$.

SOLUTION. By the Law of Cosines applied to $\triangle COB$ we have



$$x^2 = b^2 + c^2 - 2bc \cos \angle COB$$

and hence

$$\cos \angle COB = \frac{b^2 + c^2 - x^2}{2bc}$$

Furthermore, Area $\triangle AOD = (1/2)ad \sin \angle AOD$. But $\angle AOB$ is a right angle, so $\angle AOD$ and $\angle COB$ are complementary. Thus $\sin \angle AOD = \cos \angle COB$ and

$$\text{Area}\,\triangle AOD = \frac{1}{2} ad \cos \angle COB = \frac{ad}{4bc}(b^2 + c^2 - x^2)$$

3. Let x and y be positive real numbers satisfying the inequality $x^3 + y^3 \le x - y$. Prove that $x^2 + y^2 \le 1$.

SOLUTION. Now $x^3 + y^3 \leq x - y$ implies that $0 \leq y \leq x$ and $0 \leq y^3 + y \leq x - x^3$. Thus $x^3 \leq x$, so $x \leq 1$ and $0 \leq y \leq x \leq 1$. In particular, $x(x + y) \leq 1 \cdot 2 = 2$ and $xy(x + y) \leq 2y$. Finally, $(x + y)(x^2 - xy + y^2) = x^3 + y^3 \leq x - y$, so $x^2 - xy + y^2 \leq (x - y)/(x + y)$ and

$$x^2 + y^2 \leq \frac{x - y}{x + y} + xy = \frac{x - y + xy(x + y)}{x + y} \leq \frac{x - y + 2y}{x + y} = 1$$

4. To each nonnegative integer n, we associate a new integer $f(n)$. Suppose that $f(0) = 0$, $f(1) = 1$, and that for $n \geq 2$ we have $f(n) - 2f(n - 1) + f(n - 2) = (-1)^n(2n - 4)$. Describe $f(n)$ in terms of n.

SOLUTION. Since $f(n) = 2f(n - 1) - f(n - 2) + (-1)^n(2n - 4)$ for $n \geq 2$, we can compute $f(n)$ for some additional small values of n. For example

$$\begin{aligned}
&f(0) = 0,\ \ f(1) = 1,\\
&f(2) = 2f(1) - f(0) + (4 - 4) = 2,\\
&f(3) = 2f(2) - f(1) - (6 - 4) = 1,\\
&f(4) = 2f(3) - f(2) + (8 - 4) = 4,\\
&f(5) = 2f(4) - f(3) - (10 - 4) = 1
\end{aligned}$$

In general, it looks like $f(n) = 1$ if n is odd, and $f(n) = n$ if n is even. We verify this by mathematical induction. Namely, we know that the statement is true for small values of n. Now suppose it is true for $n = 0, 1, 2, \ldots, k-1$ and use $f(k) = 2f(k-1) - f(k-2) + (-1)^k(2k-4)$. If k is odd, then $f(k-1) = k-1$ and $f(k-2) = 1$, so $f(k) = 2(k-1) - 1 - (2k-4) = 1$. On the other hand, if k is even, then $f(k-1) = 1$ and $f(k-2) = k-2$, so $f(k) = 2 - (k-2) + (2k-4) = k$. Thus, in either case, $f(k)$ has the correct value, so the statement is now true for $n = 0, 1, 2, \ldots, k - 1, k$. Continuing in this manner, we see that the statement holds for all n.

5. Let us define a process which replaces each 4-tuple of nonzero real numbers $t = (a, b, c, d)$ by a new 4-tuple $t' = (a', b', c', d')$ where $a' = ab$, $b' = bc$, $c' = cd$, and $d' = da$. Suppose we start with a 4-tuple and apply this process again and again. Show that if we ever return to the original 4-tuple, then we must have started with (1, 1, 1, 1).

SOLUTION. If $t = (a, b, c, d)$, let $p(t) = abcd$ be the product of the four nonzero real numbers which make up the 4-tuple. Notice that $p(t') = a'b'c'd' = (ab)(bc)(cd)(da) = (abcd)^2 = p(t)^2$. Now let us start with $t_0 = (a_0, b_0, c_0, d_0)$, let $t_1 = t_0'$, let $t_2 = t_1'$, and so on. Then the above formula implies that $p(t_1) = p(t_0)^2$, $p(t_2) = p(t_1)^2 = p(t_0)^{2^2}$, $p(t_3) = p(t_2)^2 = p(t_0)^{2^3}$, and in general that $p(t_k) = p(t_0)^{2^k}$. In particular, if $t_n = t_0$ for some $n \geq 1$, then $p(t_0) = p(t_n) = p(t_0)^{2^n}$, and hence $p(t_0) = 1$ since 2^n is even. Therefore, $p(t_k) = 1$ for all k, and we write $t_k = (a_k, b_k, c_k, d_k)$.

Note that, if $t = (a, b, c, d)$ and $p(t) = 1$, then $a'c' = (ab)(cd) = 1$ and $b'd' = 1$. Thus, $a_k c_k = 1 = b_k d_k$ for all $k \geq 1$. But, when $k = n$ we are back to t_0, so $a_k c_k = 1 = b_k d_k$ for all $k \geq 0$. Finally, since $a_0 b_0 = 1$, we have $a_2 = a_1 b_1 = (a_0 b_0)(b_0 c_0) = b_0^2$ and similarly $b_2 = c_0^2$, $c_2 = d_0^2$ and $d_2 = a_0^2$. Continuing in this manner yields $a_8 = a_0^8$, $b_8 = b_0^8$, $c_8 = c_0^8$, $d_8 = d_0^8$ and therefore

$$a_{8k} = a_0^{8^k},\ b_{8k} = b_0^{8^k},\ c_{8k} = c_0^{8^k},\ d_{8k} = d_0^{8^k}$$

But $t_n = t_0$ implies that $t_{8n} = t_0$, so $a_0 = a_{8n} = a_0^{8^n}$ and hence $a_0 = 1$ since 8^n is even. Similarly, $b_0 = c_0 = d_0 = 1$ and $t_0 = (1, 1, 1, 1)$.