WISCONSIN MATHEMATICS SCIENCE & ENGINEERING TALENT SEARCH
SOLUTIONS TO PROBLEM SET I (1994-95)

1. Let \(a + 0d, \ a + 1d, \ a + 2d, \ a + 3d, \ldots\) be the arithmetic progression determined by the positive integers \(a\) and \(d\). Show that this sequence either contains no perfect squares or it contains infinitely many perfect squares.

**SOLUTION.** Assume that the arithmetic progression contains the perfect square \(x^2\) for some positive integer \(x\), and say \(a + yd = x^2\) for some integer \(y\). Then for any positive integer \(n\),

\[
(x + nd)^2 = x^2 + 2nd + d^2 = a + yd + 2nd + d^2 = a + (y + 2n + d)d
\]

belongs to the progression, and therefore the sequence contains infinitely many squares.

2. Suppose that the diagonals \(AC\) and \(BD\) each divide the quadrilateral \(ABCD\) into two triangles of equal area. Prove that \(ABCD\) is a parallelogram.

**SOLUTION.** Let the two diagonals intersect at the point \(O\) and label the lengths of the line segments so that \(OA\) has length \(a\), \(OB\) has length \(b\), etc. By assumption,

\[
\text{Area}(\triangle ABC) = \text{Area}(\triangle ADC) = (1/2)\text{Area}(ABCD)
\]

\[
\text{Area}(\triangle BAD) = \text{Area}(\triangle BCD) = (1/2)\text{Area}(ABCD)
\]

so all four triangles have the same area. But

\[
\text{Area}(\triangle ABC) = \text{Area}(\triangle OAB) + \text{Area}(\triangle OBC)
\]

\[
\text{Area}(\triangle BCD) = \text{Area}(\triangle OCD) + \text{Area}(\triangle OBC)
\]

so \(\text{Area}(\triangle OAB) = \text{Area}(\triangle OCD)\). Now note that \(\text{Area}(\triangle OAB) = (1/2)ab \sin \angle AOB\) and that \(\text{Area}(\triangle OCD) = (1/2)cd \sin \angle COD\), so it follows that \(ab = cd\) since \(\angle AOB = \angle COD\). Similarly \(bc = ad\), and therefore \((ab)(bc) = (cd)(ad)\), so \(b^2 = d^2\) and \(b = d\). In the same way, we get \(a = c\), and it follows by SAS that \(\triangle OAD \cong \triangle OCB\). Thus \(\angle OCB = \angle OAD\) and we conclude that \(AD \parallel BC\). Similarly, we can prove that \(AB \parallel CD\).

3. Prove that

\[
(x + y + z + w)^2 \geq (8/3)(xy + xz + xw + yz + yw + zw)
\]

for all real numbers \(x, \ y, \ z, \) and \(w\).

**SOLUTION.** Since

\[
3(x^2 + y^2 + z^2 + w^2) - 2(xy + xz + xw + yz + yw + zw)
\]

\[
= (x - y)^2 + (x - z)^2 + (x - w)^2 + (y - z)^2 + (y - w)^2 + (z - w)^2 \geq 0
\]

it follows that

\[
x^2 + y^2 + z^2 + w^2 \geq (2/3)(xy + xz + xw + yz + yw + zw)
\]
Now let us start with $t$ having exponent divisible by 7. In other words, $m$ and therefore $p$ is a prime factor of $t$, and we have $m = p^n$ where $x + u = 7a$ and $yv = b$. But if $p$ divides both $m$ and $m + 1$, then $p$ divides $(m + 1) - m = 1$, and this is not true. Therefore either $x$ or $u$ is zero and hence either $m = p^7a$, $m + 1 = v$ or $m = y$, $m + 1 = p^7a v$.

By considering all the prime factors of $t$, we conclude that each prime factor of $m$ or $m + 1$ has exponent divisible by 7. In other words, $m$ and $m + 1$ are both 7th powers of integers, say $m = r^7$ and $m + 1 = s^7$. Finally, we have $s > r$, so $s \geq r + 1$ and therefore

$$m + 1 = s^7 \geq (r + 1)^7 \geq r^7 + 7r^6 \geq m + 7$$

a contradiction. Thus $m(m + 1)$ is not a 7th power.

5. Let us define a process which replaces each triple of real numbers $t = (a, b, c)$ by a new triple $t' = (a', b', c')$ where $a' = a + b$, $b' = b + c$ and $c' = c + a$. Suppose we start with a triple and apply this process again and again. Show that if we ever return to the original triple, then we will return after just six steps.

**SOLUTION.** If $t = (a, b, c)$, let $s(t) = a + b + c$ be the sum of the three real numbers which make up the triple. Notice that

$$s(t') = a' + b' + c' = (a + b) + (b + c) + (c + a) = 2(a + b + c) = 2s(t)$$

Now let us start with $t_0 = (a_0, b_0, c_0)$, let $t_1 = t_0'$, let $t_2 = t_1'$, and so on. Then the above formula implies that $s(t_1) = 2s(t_0)$, $s(t_2) = 2s(t_1) = 2^2s(t_0)$, $s(t_3) = 2s(t_2) = 2^3s(t_0)$, and in general that $s(t_n) = 2^n s(t_0)$. In particular, if $t_n = t_0$ for some $n \geq 1$, then $s(t_0) = s(t_n) = 2^n s(t_0)$, and hence $s(t_0) = 0$.

We have shown that if $t_n = t_0$ for some $n \geq 1$, then $s(t_0) = 0$. We now show that $s(t_0) = 0$ implies that we return to the original triple in six steps. Since $0 = s(t_0) = a_0 + b_0 + c_0$, we have

$$t_1 = (a_0 + b_0, b_0 + c_0, c_0 + a_0)$$

and

$$t_2 = (a_0 + b_0 + b_0 + c_0, b_0 + c_0 + c_0 + a_0, c_0 + a_0 + a_0 + b_0) = (b_0, c_0, a_0)$$

Similarly, we get $t_4 = (c_0, a_0, b_0)$ and then $t_6 = (a_0, b_0, c_0) = t_0$. 

and therefore

$$(x + y + z + w)^2 = x^2 + y^2 + z^2 + w^2 + 2(xy + xz + xw + yz + yw + zw) \geq (8/3)(xy + xz + xw + yz + yw + zw)$$

4. If $m$ is a positive integer, can $m(m + 1)$ be the 7th power of an integer?

**SOLUTION.** The answer is “no”. Suppose that $m(m + 1) = t^7$ for some integer $t$, and write $t$ as a product of primes to powers. If $p$ is a prime and $t = p^n b$, where $b$ is the product of the remaining primes to powers, then $m(m + 1) = t^7 = p^7a b^7$. Thus $m = p^x y$ and $m + 1 = p^u v$ where $x + u = 7a$ and $yv = b$. But if $p$ divides both $m$ and $m + 1$, then $p$ divides $(m + 1) - m = 1$, and this is not true. Therefore either $x$ or $u$ is zero and hence either $m = p^7a$, $m + 1 = v$ or $m = y$, $m + 1 = p^7a v$.