WISCONSIN MATHEMATICS SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET I (1994-95)

1. Let $a + 0d,\ a + 1d,\ a + 2d,\ a + 3d,\ \ldots$ be the arithmetic progression determined by the positive integers a and d. Show that this sequence either contains no perfect squares or it contains infinitely many perfect squares.

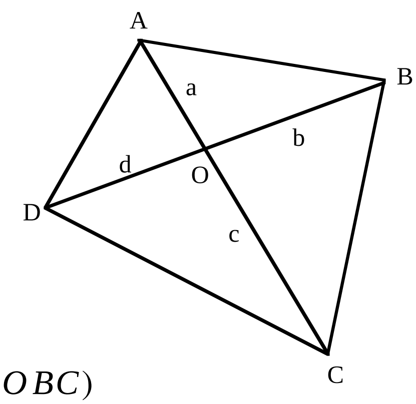
SOLUTION. Assume that the arithmetic progression contains the perfect square x^2 for some positive integer x, and say $a + yd = x^2$ for some integer y. Then for any positive integer n,

$$(x + nd)^2 = x^2 + 2nd + d^2 = a + yd + 2nd + d^2 = a + (y + 2n + d)d$$

belongs to the progression, and therefore the sequence contains infinitely many squares.

2. Suppose that the diagonals AC and BD each divide the quadrilateral $ABCD$ into two triangles of equal area. Prove that $ABCD$ is a parallelogram.

SOLUTION. Let the two diagonals intersect at the point O and label the lengths of the line segments so that OA has length a, OB has length b, etc. By assumption,



$$\text{Area}(\triangle ABC) = \text{Area}(\triangle ADC) = (1/2)\text{Area}(ABCD)$$
$$\text{Area}(\triangle BAD) = \text{Area}(\triangle BCD) = (1/2)\text{Area}(ABCD)$$

so all four triangles have the same area. But

$$\text{Area}(\triangle ABC) = \text{Area}(\triangle OAB) + \text{Area}(\triangle OBC)$$
$$\text{Area}(\triangle BCD) = \text{Area}(\triangle OCD) + \text{Area}(\triangle OBC)$$

so $\text{Area}(\triangle OAB) = \text{Area}(\triangle OCD)$. Now note that $\text{Area}(\triangle OAB) = (1/2)ab\sin\angle AOB$ and that $\text{Area}(\triangle OCD) = (1/2)cd\sin\angle COD$, so it follows that $ab = cd$ since $\angle AOB = \angle COD$. Similarly $bc = ad$, and therefore $(ab)(bc) = (cd)(ad)$, so $b^2 = d^2$ and $b = d$. In the same way, we get $a = c$, and it follows by SAS that $\triangle OAD \cong \triangle OCB$. Thus $\angle OCB = \angle OAD$ and we conclude that $AD \parallel BC$. Similarly, we can prove that $AB \parallel CD$.

3. Prove that

$$(x + y + z + w)^2 \geq (8/3)(xy + xz + xw + yz + yw + zw)$$

for all real numbers $x,\ y,\ z$, and w.

SOLUTION. Since

$$\begin{aligned}3(x^2 + y^2 + z^2 + w^2) &- 2(xy + xz + xw + yz + yw + zw)\\ &= (x - y)^2 + (x - z)^2 + (x - w)^2 + (y - z)^2 + (y - w)^2 + (z - w)^2 \geq 0\end{aligned}$$

it follows that

$$x^2 + y^2 + z^2 + w^2 \geq (2/3)(xy + xz + xw + yz + yw + zw)$$

and therefore

$$\begin{aligned}(x+y+z+w)^2 &= x^2+y^2+z^2+w^2+2(xy+xz+xw+yz+yw+zw)\\ &\geq (8/3)(xy+xz+xw+yz+yw+zw)\end{aligned}$$

4. If m is a positive integer, can $m(m+1)$ be the 7th power of an integer?

SOLUTION. The answer is "no". Suppose that $m(m+1)=t^7$ for some integer t, and write t as a product of primes to powers. If p is a prime and $t=p^a b$, where b is the product of the remaining prime factors, then $m(m+1)=t^7=p^{7a}b^7$. Thus $m=p^x y$ and $m+1=p^u v$ where $x+u=7a$ and $yv=b$. But if p divides both m and $m+1$, then p divides $(m+1)-m=1$, and this is not true. Therefore either x or u is zero and hence either $m=p^{7a}y,\ m+1=v$ or $m=y,\ m+1=p^{7a}v$.

By considering all the prime factors of t, we conclude that each prime factor of m or $m+1$ has exponent divisible by 7. In other words, m and $m+1$ are both 7th powers of integers, say $m=r^7$ and $m+1=s^7$. Finally, we have $s>r$, so $s\geq r+1$ and therefore

$$m+1=s^7\geq (r+1)^7\geq r^7+7r^6\geq m+7$$

a contradiction. Thus $m(m+1)$ is not a 7th power.

5. Let us define a process which replaces each triple of real numbers $t=(a,b,c)$ by a new triple $t'=(a',b',c')$ where $a'=a+b,\ b'=b+c$ and $c'=c+a$. Suppose we start with a triple and apply this process again and again. Show that if we ever return to the original triple, then we will return after just six steps.

SOLUTION. If $t=(a,b,c)$, let $s(t)=a+b+c$ be the sum of the three real numbers which make up the triple. Notice that

$$s(t')=a'+b'+c'=(a+b)+(b+c)+(c+a)=2(a+b+c)=2s(t)$$

Now let us start with $t_0=(a_0,b_0,c_0)$, let $t_1=t_0'$, let $t_2=t_1'$, and so on. Then the above formula implies that $s(t_1)=2s(t_0)$, $s(t_2)=2s(t_1)=2^2s(t_0)$, $s(t_3)=2s(t_2)=2^3s(t_0)$, and in general that $s(t_n)=2^ns(t_0)$. In particular, if $t_n=t_0$ for some $n\geq 1$, then $s(t_0)=s(t_n)=2^ns(t_0)$, and hence $s(t_0)=0$.

We have shown that if $t_n=t_0$ for some $n\geq 1$, then $s(t_0)=0$. We now show that $s(t_0)=0$ implies that we return to the original triple in six steps. Since $0=s(t_0)=a_0+b_0+c_0$, we have $t_1=(a_0+b_0,b_0+c_0,c_0+a_0)$ and

$$t_2=(a_0+b_0+b_0+c_0,b_0+c_0+c_0+a_0,c_0+a_0+a_0+b_0)=(b_0,c_0,a_0)$$

Similarly, we get $t_4=(c_0,a_0,b_0)$ and then $t_6=(a_0,b_0,c_0)=t_0$.