WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET V (2014-2015)

1. Let $x_1, x_2, \ldots, x_{12}$ be positive numbers. Show that at least one of the following must be true:

$$\frac{x_1}{x_2} + \frac{x_3}{x_4} + \frac{x_5}{x_6} + \frac{x_7}{x_8} + \frac{x_9}{x_{10}} \geq 5, \quad \frac{x_{11}}{x_{12}} + \frac{x_2}{x_1} + \frac{x_4}{x_3} + \frac{x_6}{x_5} \geq 4, \quad \text{or} \quad \frac{x_8}{x_7} + \frac{x_{10}}{x_9} + \frac{x_{12}}{x_{11}} \geq 3.$$

SOLUTION. We start by noting that for any positive number y we have $y + \frac{1}{y} \geq 2$. This can be shown, for example, using

$$0 \leq \left(\sqrt{y} - \frac{1}{\sqrt{y}}\right)^2 = y - 2 + \frac{1}{y}$$

and rearranging the terms. (This is also a direct consequence of the Arithmetic Mean/Geometric Mean Inequality.) Suppose, for the sake of contradiction, that

$$\frac{x_1}{x_2} + \frac{x_3}{x_4} + \frac{x_5}{x_6} + \frac{x_7}{x_8} + \frac{x_9}{x_{10}} < 5, \qquad \frac{x_{11}}{x_{12}} + \frac{x_2}{x_1} + \frac{x_4}{x_3} + \frac{x_6}{x_5} < 4, \quad \text{and} \quad \frac{x_8}{x_7} + \frac{x_{10}}{x_9} + \frac{x_{12}}{x_{11}} < 3.$$

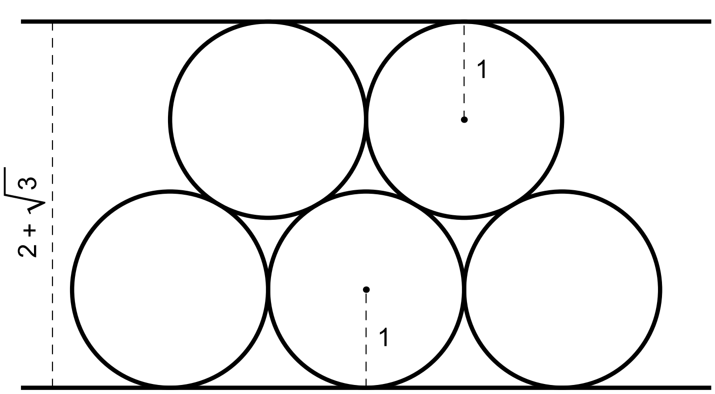
Adding the three inequalities we have that

$$\frac{x_1}{x_2} + \frac{x_2}{x_1} + \frac{x_3}{x_4} + \frac{x_4}{x_3} + \frac{x_5}{x_6} + \frac{x_6}{x_5} + \frac{x_7}{x_8} + \frac{x_8}{x_7} + \frac{x_9}{x_{10}} + \frac{x_{10}}{x_9} + \frac{x_{11}}{x_{12}} + \frac{x_{12}}{x_{11}} < 12.$$

But each of the six pairs of reciprocals, e.g. $\frac{x_1}{x_2} + \frac{x_2}{x_1}$, add to at least 2, which is a contradiction. Thus one of the three given inequalities must hold.

2. There is an infinite region bounded by two parallel lines containing infinitely many non-overlapping (possibly tangent) circles of radius 1. Suppose that every line perpendicular to the boundary lines intersects or is tangent to at least two of the circles. Find the minimum possible distance between the boundary lines. (Make sure you prove that the minimum you claim is achievable, and that there is no possible smaller width.)

SOLUTION. The minimum width is $2 + \sqrt{3}$. This can be achieved by centering circles at $(2n, 1)$ for every integer n and at $(2n + 1, 1 + \sqrt{3})$ for every integer n.



The two boundary lines are $y=0$ and $y=2+\sqrt{3}$. All of the lower circles lie between $y=0$ and $y=2$, and all of the upper circles lie between $y=\sqrt{3}$ and $y=2+\sqrt{3}$, so all the circles are in the region between the two lines. Since the centers of the lower circles are all distance 2 apart, none of the lower circles overlap, and the same argument can be made for the upper circles. The distance between the centers of an upper circle at $(2m+1, 1+\sqrt{3})$ and a lower circle at $(2n, 1)$ is

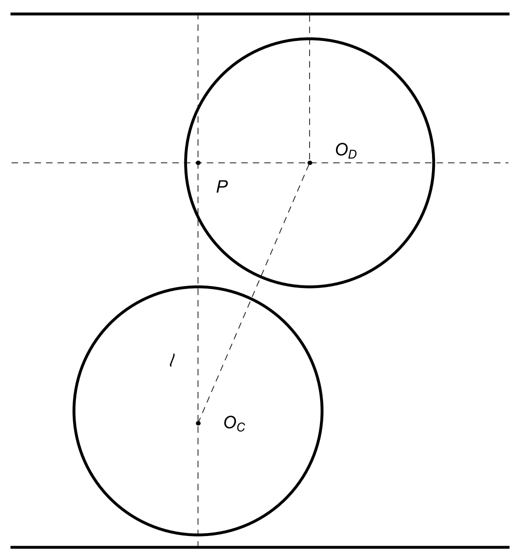
$$\sqrt{(2m+1-2n)^2+\sqrt{3}^2} \geq \sqrt{1+3} \geq 2,$$

so no upper circle intersects a lower circle. Thus we have non-overlapping circles. Finally, for any a, the line $x=a$ intersects the circles whose center are within one unit of this line, i.e. the circles with centers whose x-coordinate is in the interval $[a-1, a+1]$. This interval contains at least two integers (since its length is 2), and in our construction there is a circle with x-coordinate n for each integer n. Thus the line $x=a$ will intersect at least two of the circles (three if a is integer).

To prove that there is no smaller possible width between the parallel lines, consider one circle C with center O_C. The line ℓ through O_C and perpendicular to the boundary lines must pass through another circle, D, with center O_D. Let P be the intersection of the line ℓ and the line through O_D parallel to the boundary lines. Since circles C and D do not overlap, we have that $O_CO_D \geq 2$. Since the line ℓ intersects circle D, we have $O_DP \leq 1$. Thus

$$O_CP = \sqrt{O_CO_D^2 - O_DP^2} \geq \sqrt{3}.$$

Since the centers of the circles C and D are at least distance 1 from the boundary lines, it follows that the boundary lines are at least $2+\sqrt{3}$ apart.



3. Find all right angle triangles where the hypotenuse has length $\sqrt{2}\cdot 2^{2015}$, and the other two sides have integer lengths.

SOLUTION. Denote the lengths of the other two sides by a and b. Then by the Pythagorean theorem we have $a^2+b^2=(\sqrt{2}\cdot 2^{2015})^2=2^{4031}$. We will need to find the positive integer solutions of this equation.

Suppose that a, b are positive integer solutions to the equation. Let k be the largest non-negative integer so that 2^k divides both a and b. Then $a=2^k a_1$, $b=2^k b_1$, where a_1, b_1 are positive integers and at least one of them is odd. (Otherwise 2^{k+1} would divide both a and b.) Then we have

$$2^{4031}=a^2+b^2=2^{2k}(a_1^2+b_1^2).$$

Since the right side is divisible by 2^{2k}, we must have $2k \leq 4031$ and $k \leq 2015$. Moreover we have $2^{4031-2k}=a_1^2+b_1^2$.

The square of an even number is divisible by 4, and since $(2\ell+1)^2=4\ell(\ell+1)+1$, the square of an odd number has remainder 1 modulo 4. Thus $a_1^2+b_1^2$ must give remainder 2 modulo 4 (if both a_1 and b_1 are odd) or remainder 1 (if one is even and the other is odd), and thus it cannot be divisible by 4. If $k < 2015$, then $2^{4031-2k}$ is divisible by 4. Since $k \le 2015$, this shows that we must have $k = 2015$ and $2 = a_1^2 + b_1^2$. The only solution among positive integers is $a_1 = b_1 = 1$ (all other choices would give a larger $a_1^2 + b_1^2$), which proves that the only solution to the original equation is $a = b = 2^{2015}$.

4. Three French farmers, four English farmers, and five Spanish farmers are all attending the International Farmers Conference. They will sit in twelve chairs numbered 1 to 12 equally spaced around a round table. We have the job of assigning seats to the twelve participants. In how many different ways can we make these assignments so that each French farmer has an English farmer sitting immediately to her right.

SOLUTION. We can generate appropriate seat assignments using the following procedure:
1) We first choose the 'right-side neighbor' for each French farmer. We can do this $4 \cdot 3 \cdot 2$ different ways: the right-side neighbor for the first French farmer can be any of the 4 English farmers, then we will have 3 choices for the second and 2 choices for the third French farmer (regardless of our previous choices).
2) One of the English farmers has not been chosen in the previous step, we will sit her down first at the table. (This can be done 12 different ways.)
3) We will now seat the remaining 11 farmers: the three French-English pairs and the five Spanish farmers. Considering these as eight units (a unit is either a French-English pair or a Spanish farmer), we will sit them down one-by-one clockwise starting to the right of the already seated English farmer. (When we seat a French-English pair, the English farmer will sit to the right of her French partner.) There are 8 choices for which unit will be seated first, immediately to the right of the seated English farmer. Then we can choose the next one 7 ways, and so on, giving $8 \cdot 7 \cdots 2 \cdot 1 = 8!$ different ways of seating the remaining 11 farmers.

At the end of this procedure we have an appropriate seating assignment, and each possible seating assignment can be uniquely obtained this way. Multiplying the number of choices we make in the three steps (since in each step the number of choices do not depend on our previous decisions), we get that there are $4 \cdot 3 \cdot 2 \cdot 12 \cdot 8! = 11{,}612{,}160$ possible seating arrangements.

5. Find the value $f(2015)$, if we know that the function f is defined on positive integers and satisfies

$$f(n) = \begin{cases} n - 10, & \text{if } n > 10000, \\ f(f(n+11)), & \text{if } n \le 10000. \end{cases}$$

SOLUTION. We have

$$f(10000) = f(f(10011)) = f(10001) = 9991,$$

since $f(10011) = 10011 - 10$ and $f(10001) = 10001 - 10$. Similarly,

$$f(9999) = f(f(10010)) = f(10000) = 9991.$$

We will proceed by induction. Assume that for some $1 \le n < 10000$ it is true that $f(n+1) = f(n+2) = \cdots = f(10000) = 9991$. We will show that $f(n) = 9991$ must hold as well. This will prove that $f(n) = 9991$ for all $1 \le n \le 10000$, and thus $f(2015) = 9991$.

We will consider first the case when $n+11 > 10000$. Then $f(n) = f(f(n+11)) = f(n+1)$ which is equal to 9991 by our assumption. This proves that for $10000 - 11 < n < 10000$ we have $f(n) = 9991$.

Now consider the case when $n+11 \leq 10000$ (again with the extra assumption that $f(n+1) = f(n+2) = \cdots = f(10000) = 9991$). In this case we have $f(n) = f(f(n+11))$. Since $n+11 < 10000$, by our assumption we have $f(n+11) = 9991$ which gives $f(n) = f(9991)$. But $10000 - 11 < 9991 < 10000$, so we already know that $f(9991) = 9991$ which implies $f(n) = f(9991) = 9991$. This finishes the proof.