WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET IV (2014-2015)

1. Suppose that $x > 0$. Show that

$$\frac{2015}{(x+1)(x+4030)} \le \frac{1}{x+1} - \frac{1}{x+2} + \frac{1}{x+3} - \frac{1}{x+4} + - \cdots + \frac{1}{x+4029} - \frac{1}{x+4030}.$$

SOLUTION. We will show that for a positive integer n the following inequality holds:

$$\frac{n}{(x+1)(x+2n)} \le \frac{1}{x+1} - \frac{1}{x+2} + \frac{1}{x+3} - \frac{1}{x+4} + - \cdots + \frac{1}{x+2n-1} - \frac{1}{x+2n}.$$

For $n = 1$ this is an identity:

$$\frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}.$$

We will show the general statement by induction. We will prove that by changing n to $n+1$ in the inequality the left hand side increases by a smaller amount than the right hand side. For this we need to show

$$\frac{n+1}{(x+1)(x+2n+2)} - \frac{n}{(x+1)(x+2n)} \le \frac{1}{x+2n+1} - \frac{1}{x+2n+2} = \frac{1}{(x+2n+1)(x+2n+2)}.$$

Rearranging the terms:

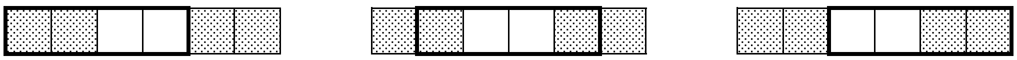
$$\begin{aligned} 0 &\le \frac{n}{(x+1)(x+2n)} - \frac{n}{(x+1)(x+2n+2)} + \frac{1}{(x+2n+1)(x+2n+2)} - \frac{1}{(x+1)(x+2n+2)} \\ &= \frac{2n}{(x+1)(x+2n)(x+2n+2)} - \frac{2n}{(x+1)(x+2n+1)(x+2n+2)} \\ &= \frac{2n}{(x+1)(x+2n)(x+2n+1)(x+2n+2)}. \end{aligned}$$

Since $x > 0$, this inequality is true. Because our steps were reversible, this proves the original inequality. Choosing $n = 2015$ finishes the proof.

2. We have a cubic box whose interior dimensions are $6 \times 6 \times 6$. Clearly, we can fill it completely with 216 unit cubes. Is there an arrangement of 27 rectangular blocks with dimensions $1 \times 2 \times 4$ that exactly fills the cubic box? You can assume that in such an arrangement each rectangular block would cover exactly eight of the smaller unit cubes.

SOLUTION. We will show that it is impossible to do that. We first divide the box into 27 cubes, each $2 \times 2 \times 2$, and color them alternatingly white and black so that no black cube shares a face with a white cube (in a 3 dimensional “checkerboard” pattern). There are different numbers of white cubes and black cubes since 27 is odd. Now divide each $2 \times 2 \times 2$ cube further into eight $1 \times 1 \times 1$ cubes. Then the numbers of the small black and small white unit cubes are also different.

Each $1 \times 2 \times 4$ block will occupy exactly eight of the small cubes. We will show that four of these must be white and four must be black. Consider a $1 \times 1 \times 6$ column of the cubic box in a direction parallel to one of the sides. It will have two black, two white, two black or two white, two black, two white cubes in that order.



It is easy to check that if we place a $1 \times 1 \times 4$ block into this column, then it will have exactly two black and two white small cubes. (See the picture above.) But a $1 \times 2 \times 4$ block occupies exactly two $1 \times 1 \times 4$ blocks, and each one must be inside a $1 \times 1 \times 6$ column of the box. Thus a $1 \times 2 \times 4$ block has to cover four white and four black small cubes.

If we could fill the box with $1 \times 2 \times 4$ blocks, then this would mean that there are the same number of black and white small cubes. But we know that this is not the case, thus we showed that it is impossible to fill in the big box.

3. We choose three points in a square of unit side length and measure the distances between each pair of the points. Find the maximal possible value of the *minimum* of these three numbers.

SOLUTION. We will show that the minimum of the three distances is always at most $\sqrt{6} - \sqrt{2} \approx 1.035$.

First we show that this value can be achieved. Denote the vertices of the square by A, B, C, D and let X and Y be points on the side BC and CD with $\overline{BX} = \overline{YD} = 2 - \sqrt{3}$. Then by the Pythagorean theorem the points A, X, Y form a regular triangle with side length:

$$\overline{AX}^2 = \overline{AY}^2 = 1 + (2 - \sqrt{3})^2 = 8 - 2\sqrt{3}$$
$$\overline{XY}^2 = 2(1 - (2 - \sqrt{3}))^2 = 8 - 2\sqrt{3}.$$

We can also check that $\sqrt{8 - 2\sqrt{3}} = \sqrt{6} - \sqrt{2}$ by squaring both sides.

Our next task is to show that if we choose any three points P, Q, R in the square, then the smallest of the three distances is at most $\sqrt{6} - \sqrt{2}$. We first show that we can assume that all three points are on the boundary of the square. To see this, consider a triangle XYZ with acute angles at X and Y and assume that the altitude dropped from Z intersects XY at point H. Now imagine that we choose points X', Y', Z' which are on the rays HX, HY, HZ, respectively with $\overline{X'H} \geq \overline{XH}$, $\overline{Y'H} \geq \overline{YH}$ and $\overline{Z'H} \geq \overline{ZH}$. (See picture below.) Then the distances between the points X', Y', Z' are at least as large as the corresponding distances between the points X, Y, Z. E.g.

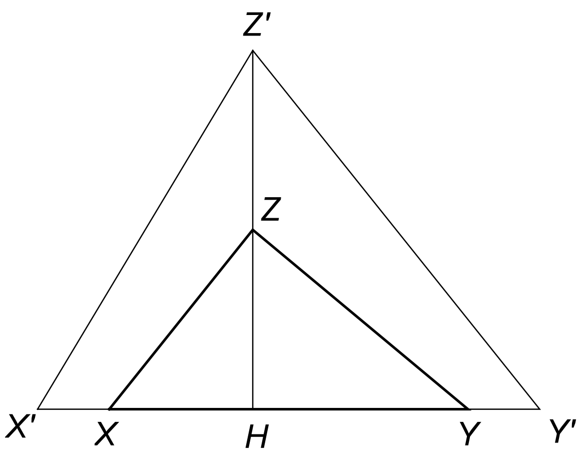
$$\overline{X'Z'} = \sqrt{\overline{X'H}^2 + \overline{Z'H}^2} \geq \sqrt{\overline{XH}^2 + \overline{ZH}^2} = \overline{XZ}.$$

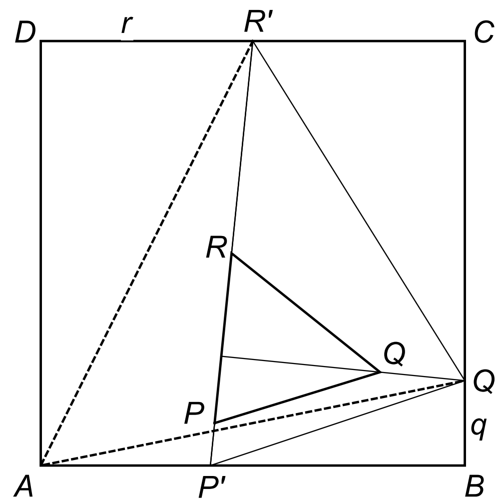
Thus, by moving the points X, Y, Z on the rays HX, HY, HZ ‘away’ from H we will not decrease the minimum distance among these points. (This is also true if Z is on the line segment XY, i.e. $H = Z$.)

Now if we have three points P, Q, R inside the square then at least two of the angles of the triangle PQR are acute (or the three points are on a line), so repeating the previous procedure we can move the points P, Q, R onto the boundary of the square to get P', Q', R', in a way that the minimal distance among the new points is at least as large as the minimal distance among the old ones. (See picture below.)

Thus it is enough to consider the case when we have three points P', Q', R' are on the boundary of the square, and we have to prove that one of the distances among them is at most $\sqrt{6}-\sqrt{2}$. If any two of these points are on the same side of the square then their distance is at most $1 < \sqrt{6}-\sqrt{2}$, so we may assume that the three points are on three different sides. Let P' be on AB, Q' on BC and R' on CD. Assume that $\overline{AP'} \leq \overline{DR'}$ (the proof will work similarly if the opposite inequality holds). Then moving P' to A will not decrease the distances. Indeed, $\overline{Q'R'}$ will not change and

$$\overline{AQ'} = \sqrt{1+\overline{Q'B}^2} \geq \sqrt{\overline{P'B}^2+\overline{Q'B}^2} = \overline{P'Q'}, \quad \overline{AR'} = \sqrt{1+\overline{DR'}^2} \geq \sqrt{1+(\overline{DR'}-\overline{AP'})^2} = \overline{P'R'}.$$





Let $\overline{BQ'} = q$ and $\overline{DR'} = r$. If we assume that $\overline{AQ'}, \overline{AR'}$ and $\overline{Q'R'}$ are all bigger than $\sqrt{6}-\sqrt{2}$ then

$$\sqrt{1+q^2} > \sqrt{6}-\sqrt{2}, \quad \sqrt{1+p^2} > \sqrt{6}-\sqrt{2}, \quad \sqrt{(1-p)^2+(1-q)^2} > \sqrt{6}-\sqrt{2}.$$

From the first inequality we get

$$q^2 > (\sqrt{6}-\sqrt{2})^2 - 1 = 7 - 4\sqrt{3} = (2-\sqrt{3})^2$$

and $q > 2-\sqrt{3}$. Similarly, $p > 2-\sqrt{3}$. But then

$$(1-p)^2 + (1-q)^2 < 2(\sqrt{3}-1)^2 = (\sqrt{6}-\sqrt{2})^2$$

which leads to a contradiction. This shows that after moving the points around in a way that the minimum distance did not decrease, we arrived to a configuration where the minimum distance is at most $\sqrt{6}-\sqrt{2}$. This shows that this was true for the original configuration as well.

4. Find all the four-digit numbers $\underline{a}\,\underline{b}\,\underline{c}\,\underline{d}$ which when multiplied by 4 give a product equal to the number with the digits reversed, $\underline{d}\,\underline{c}\,\underline{b}\,\underline{a}$. (The digits do not need to be different.)

SOLUTION. The value of such a number is $1000a+100b+10c+d$, so the given condition is that $4(1000a+100b+10c+d) = 1000d+100c+10b+a$. This means $3999a+390b-60c-996d = 0$, or by dividing by 3, $1333a+130b-20c-332d = 0$. Because $130b-20c-332d$ is even, the digit a

must be even, and because $4a$ must be less than 10, $a = 2$. Thus, $332d + 20c - 130b = 2666$ or by dividing by 2, $166d + 10c - 65b = 1333$. Finding the remainders of each side modulo 5 shows that $d \equiv 3 \pmod 5$, so $d = 3$ or $d = 8$. Clearly, if $a = 2$, then $d = 8$. It follows that, $10c - 65b = 5$ or $2c = 1 + 13b$. If b is bigger than 1 then $13b + 1$ would be bigger than 20, and if $b = 0$ then c would not be an integer. Hence the only way we can have $2c = 1 + 13b$ is for b to be 1 and c to be 7. Thus, the only number satisfying the given condition is 2178, with $4 \times 2178 = 8712$.

5. In the sequence $2, 0, 1, 5, 8, \ldots$ each number is the last digit of the sum of the previous four. Show that the digits 2, 0, 1, 5 will show up eventually again in this order, but the digits 2, 0, 1, 4 will never show up in this order.

SOLUTION. Denote the numbers in the sequence by $a_1, a_2, a_3 \ldots$ (with $a_1 = 2, a_2 = 0, \ldots$). If we have four consecutive numbers from the sequence: $a_k, a_{k+1}, a_{k+2}, a_{k+3}$, then we can determine both a_{k+4} and a_{k-1}, i..e. the next and the previous entry. a_{k+4} is just the last digit (or the nonnegative remainder mod 10) of $a_k + a_{k+1} + a_{k+2} + a_{k+3}$, while a_{k-1} is the nonnegative reminder mod 10 of $a_k + a_{k+1} + a_{k+2} - a_{k+3}$. (Since a_{k+3} is the remainder mod 10 of $a_{k-1} + a_k + a_{k+1} + a_{k+2}$.)

The entries in the sequence are between 0 and 9 which means that there are at most $10^4 = 10000$ possible four digit sequences we can see next to each other. This means that we will see a four digit sequence x, y, z, w that appears at least twice, and we will find $0 < k < \ell$ with

$$a_k = x, a_{k+1} = y, a_{k+2} = z, a_{k+3} = w, \quad a_\ell = x, a_{\ell+1} = y, a_{\ell+2} = z, a_{\ell+3} = w.$$

But the sequence is completely determined (both forward and backward) by any of its four consecutive entries. So if starting at $a_k, a_{k+1}, a_{k+2}, a_{k+3}$ we get the same four numbers after $\ell - k$ steps, then the whole sequence of numbers between a_k and a_ℓ will be repeated again and again both in the forward and the backward directions (such sequences are called periodic). In particular, for any $n > 0$ we will have $a_n = a_{n+(\ell-k)}$. But that means that the numbers 2, 0, 1, 5 will show up again at $a_{1+(\ell-k)}, a_{2+(\ell-k)}, a_{3+(\ell-k)}, a_{4+(\ell-k)}$.

For the second part of the problem we will consider the modulo 2 remainders of the numbers in the sequence. We denote this by b_k for the kth entry. Then

$$b_1 = 0, b_2 = 0, b_3 = 1, b_4 = 1, b_5 = 0, b_6 = 0, b_7 = 0, b_8 = 1, b_9 = 1$$

which shows that we see the same four numbers at positions 1, 2, 3, 4 and 6, 7, 8, 9. Using the same argument as before, we see that the sequence will be periodic and $b_k = b_{k+5}$ for all k. In order to see the digits 2, 0, 1, 4 in the original sequence we would need to see the numbers 0, 0, 1, 0 next to each other in the second sequence. But since we cannot see them within the first eight numbers, we can be sure that they will never appear in the sequence in this order.