WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET III (2014-2015)

1. Find all integers a such that $a^3 + a + 1$ divides $a^4 + a + 1$.

SOLUTION. We will use the notation $x|y$ if x divides y.

We can check that $a = 1, 0, -1$ satisfy the requirement. Since $a^3+a+1|a(a^3+a+1) = a^4+a^2+a$, if $a^3+a+1|a^4+a+1$, then $a^3+a+1|a^4+a^2+a-(a^4+a+1) = a^2-1$. If a^2-1 is not 0, this implies $|a^3+a+1| \leq |a^2-1|$. If a is different from $-1, 0, 1$, then $|a^2-1| = a^2-1 > 0$. If $a > 1$, $a^3 > a^2$, and so we cannot have $a^3+a+1 \leq a^2-1$. If $a < -1$, then $|a^3+a+1| = -(a^3+a+1) = (-a)^3-a-1$. In this case $(-a)^3 > a^2$, and so we cannot have $(-a)^3 - a - 1 \leq a^2 - 1$. This shows that there are no other solutions.

2. You have 5 green and 7 red balls, and two empty boxes. You place all the balls in the two boxes so that each box contains at least one ball. Your friend then chooses one of the two boxes randomly, and picks a ball randomly from the chosen box. If the chosen ball is green, then you win a prize. How should you arrange the balls between the two boxes initially to maximize the probability of winning? What is this probability?

SOLUTION. If there are g green and r red balls in one box, then we would pick a green ball with probability $\frac{g}{g+r}$ from that box. The other box would have $5-g$ green and $7-r$ red balls, so picking a ball randomly would produce a green ball with probability $\frac{5-g}{12-(r+g)}$. Since the boxes are chosen with equal probability, the probability of choosing a green ball with this arrangement of balls will be the average of the two probabilities: $\frac{1}{2}\left(\frac{g}{g+r} + \frac{5-g}{12-(r+g)}\right)$.

We have to figure out how to maximize this quantity for the possible values of r and g. These values are $0 \leq r \leq 5$ and $0 \leq g \leq 7$, with the $(r, g) = (0, 0)$ and $(5, 7)$ cases excluded. Suppose that we put a single green ball in one of the boxes and the remaining 4 green and 7 red balls in the other box. Then the probability of winning is $\frac{1}{2}\left(\frac{1}{1} + \frac{4}{11}\right) = \frac{15}{22}$. We will show that we cannot get a higher winning probability than that.

If one of the boxes contains only red balls, then the probability of winning is at most $1/2$ (since we immediately lose if we pick the all-red box), which is less than $\frac{15}{22}$. If one of the boxes contains only green balls and more than one, then moving one of them to the other box will increase the probability of winning. This is because the probability of picking a green ball increased in the 'mixed' box (by the additional green), and it did not change for the all-green box. So among configurations with an all green box, the example above with a box with just one green ball maximizes your probability of winning.

Now assume that there are red and green balls in both boxes. We assume that the first box has at most as many balls as the second one, meaning that $r + g \leq 12 - (r + g)$. (The proof will work the same way in the opposite case.) Now take a green ball from the second box, a red from the first, and switch them. Then the probability of winning changed by $\frac{1}{2}\left(\frac{1}{r+g} - \frac{1}{12-(r+g)}\right) \geq 0$, which means that it did not decrease. Repeating this step a couple of more times we either get all the greens in the first box or all the reds in the second box. But we have already checked that those cases have a winning probability at most $\frac{15}{22}$, so this must be true for our starting configuration as well. This shows that the probability of winning is at most $\frac{15}{22}$ for all possible arrangements.

3. Consider the sequence $a_0, a_1, a_2, \dots$ where $a_0 = 1$, and where $a_{k+1} = a_k + \frac{2}{a_k}$ for $k \geq 0$. (Thus $a_1 = 1 + \frac{2}{1} = 3$ and $a_2 = 3 + \frac{2}{3} = \frac{11}{3}$.) Prove that $a_{2015} > 89$.

SOLUTION. Clearly, each number in the sequence is positive, and the sequence is increasing. By squaring $a_{k+1} = a_k + \frac{2}{a_k}$ we get $a_{k+1}^2 = a_k^2 + 4 + \frac{4}{a_k^2}$ for $k \geq 0$. Using this identity repeatedly:

$$\begin{aligned}
a_{2015}^2 &= a_{2014}^2 + 4 + \frac{4}{a_{2014}^2} \\
&= (a_{2013}^2 + 4 + \frac{4}{a_{2013}^2}) + 4 + \frac{4}{a_{2014}^2} = a_{2013}^2 + 2 \cdot 4 + \frac{4}{a_{2014}^2} + \frac{4}{a_{2013}^2} \\
&= (a_{2012}^2 + 4 + \frac{4}{a_{2012}^2}) + 2 \cdot 4 + \frac{4}{a_{2014}^2} + \frac{4}{a_{2013}^2} = a_{2012}^2 + 3 \cdot 4 + \frac{4}{a_{2014}^2} + \frac{4}{a_{2013}^2} + \frac{4}{a_{2012}^2} \\
&\vdots \\
&= a_0^2 + 2015 \cdot 4 + \frac{4}{a_{2014}^2} + \frac{4}{a_{2013}^2} + \frac{4}{a_{2012}^2} + \cdots + \frac{4}{a_0^2}
\end{aligned}$$

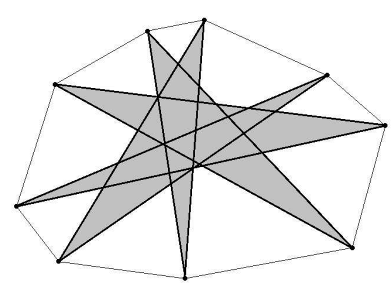
The last expression is at least as big as $a_0^2 + 2015 \cdot 4 + \frac{4}{a_0^2} = 8065 > 89^2$ which proves $a_{2015} > 89$.

4. Alice and Brittany play the following game. They start out with n marbles and take turns with Alice going first. In each turn Alice can take 1 or 2 marbles and Brittany can take 1, 3 or 5 marbles (if there are enough marbles left). The winner is the player who takes the last marble. For which n will Alice have a winning strategy?

SOLUTION. We will show that Alice can win for any n, if she plays smartly. If $n = 1$ or 2, then Alice can win in a single move by taking all the marbles. If $n = 3$ or 4, then she can take some marbles so that there are exactly 2 left on the table; in that case Brittany can only take one and Alice can take the remaining marble for the win. To deal with higher numbers, we use induction. Assume that $n \geq 5$, and we have proved that Alice can win for any initial number which is less than n. We will show that then Alice has a winning strategy for n as well.

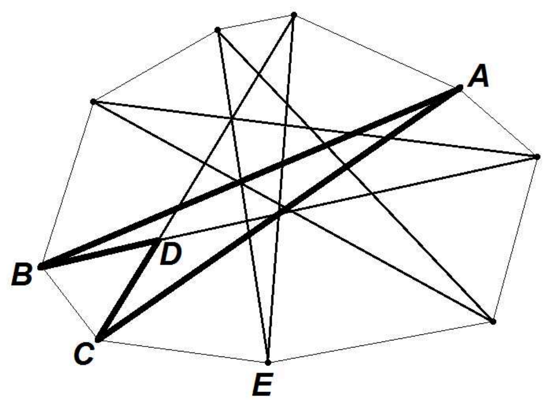
One of the numbers from $n - 1$ and $n - 2$ is not equal to any of 1, 3, or 5, which means that Alice can make a first move so that Brittany cannot win immediately afterwards. This means that Brittany's move there will be some marbles left on the table, and their number will be less than n. According to our induction hypothesis Alice will have a winning strategy for that particular number. But this gives her a winning strategy for n, which finishes the proof.

5. Let $A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8$, and A_9 be the nine vertices of any convex nonagon in that order. Draw the 9 diagonals that connect each vertex to the vertex four away from it, that is, A_1A_5, A_2A_6, A_3A_7, A_4A_8, A_5A_9, A_6A_1, A_7A_2, A_8A_3, and A_9A_4. These 9 diagonals form a nine-pointed star. (See the diagram for an example.) Show that the degree measures of the nine angles at the points of the star add to 180°.



SOLUTION. Let the sum of the measures of the angles at the points of the star be α. For each vertex A of the nonagon, let B and C be the endpoints of the diagonals drawn to A, and let D

be the intersection of the other two diagonals connected to B and C as shown in the diagram below. Consider the quadrilateral $ABDC$. Since the four internal angles of this quadrilateral have measures that add to 360°, it follows that the measure of the external $\angle BDC$ is equal to the sum of the angles at the three points of the star, $\angle A + \angle B + \angle C$. Call $\triangle BCD$ an *edge triangle*, and $\angle CBD$ and $\angle BCD$ the *base angles* of the edge triangle. Thus, the sum of the measures of the base angles of edge $\triangle BCD$ is $180° - (\angle A + \angle B + \angle C)$.



Note that the internal angle of the nonagon $\angle BCE$ consists of base angles of two adjacent edge triangles and one angle at the point of the star ($\angle C$). The nine interior angles of the nonagon are made up of the base angles of the nine edge triangles plus the sum of the measures of the angles at the points of the star. Using the arguments of the previous paragraph we see that if we add all the base angles, we get $9 \cdot 180°$ minus three times the sum of angles of the star, $9 \cdot 180° - 3\alpha$. Thus the sum of the internal angles of the nonagon is $9 \cdot 180° - 3\alpha + \alpha = 9 \cdot 180° - 2\alpha$. But we know that the measures of the internal angles of a polygon with n vertices is $(n-2) \cdot 180°$, so we get $(9-2) \cdot 180° = 9 \cdot 180° - 2\alpha$. Solving for α shows that $\alpha = 180°$.