WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET II (2014-2015)

1. Suppose that x and y are positive integers with $x+y=10^{100}$. Assume that x and y have the exact same decimal digits, only in a different order. Show that both x and y are divisible by 5.

SOLUTION. Both x and y have 100 digits. Denote the digits (starting from the unit digits) by $x_1, x_2, \ldots, x_{100}$ and $y_1, y_2, \ldots, y_{100}$. Since $x+y=10^{100}$, we must have $x_1+y_1=0$ or 10. In the first case $x_1=y_1=0$ and both x and y are divisible by 10 (and hence by 5). We will show that the second case is not possible. If $x_1+y_1=10$, then we have to 'carry the 1' and x_2+y_2 must be equal to 9 to get zero in the appropriate digit of $x+y$. But then we have to carry the 1 again, and a similar argument shows that $x_i+y_i=9$ for all $i=3,4,\ldots,100$. Then the total sum of the the digits of x and y will be $99\cdot 9+10$, an odd number. But in this sum every digit must appear an even number of times (because the digits of y are the same as the digits of x in a different order) which would make the sum even. The contradiction shows that x_1+y_1 cannot be 10, and hence both x and y must be divisible by 5.

2. In the sequence $2,3,4,6,9,14,\ldots$ each number is 1 less than the sum of the previous two terms. Show that all integers larger than 1 can be written as the sum of one or more distinct elements of the series.

SOLUTION. Denote the elements of the sequence by $a_1, a_2, \ldots$. Then we have $a_1=2, a_2=3$ and $a_{n+1}=a_n+a_{n-1}-1$ for $n>1$. It is easy to see that the sequence is increasing. We also have $a_{k+1}\le 2a_k$, since $a_{k+1}=a_k+a_{k-1}-1\le 2a_k-1$ for $k>1$ (and $a_2=3<2\cdot a_1=4$.)

Since 2 and 3 appear in the sequence, it is enough to show the statement of the problem for numbers bigger than 3. Assume that we have already shown that all numbers between 2 and n for some $n>3$ can be written as a sum of one or more distinct elements of the series. We will show that this will be true for $n+1$ as well. Since 2, 3, and 4 are in the sequence, the statement will follow from this by induction.

Let a_k be the largest element of the sequence which is not bigger than $n+1$. If $a_k=n+1$, then we are done because we can just use a_k to represent $n+1$. If $a_k=n$, then k must be at least 3 and thus $n=a_k=a_{k-1}+a_{k-2}-1$ which means that $n+1=a_{k-1}+a_{k-2}$, the sum of two distinct elements from the sequence. If $a_k<n$, then $b=n+1-a_k$ is between 2 and n and by our assumption it can be written as the sum of one or more distinct elements from the sequence. None of the entries in that sum can be equal to a_k, because then $a_k\le b$ would imply $2a_k\le n+1$ which would mean that we could have chosen $a_{k+1}\le 2a_k$ as an entry not bigger than $n+1$. But if a_k does not appear in the sum representing b, then adding a_k to that sum gives a proper representation of $n+1$, finishing the proof.

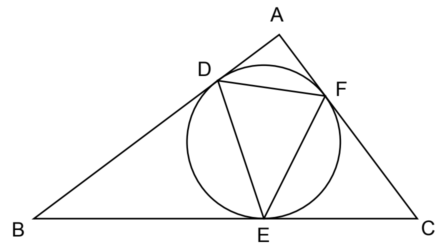
3. The incircle of $\triangle ABC$ touches the sides of the triangle in three points. Show that if the triangle determined by the three touchpoints is similar to $\triangle ABC$, then it must be equilateral.

SOLUTION. Let the incircle of $\triangle ABC$ be tangent to sides $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$ at points D, E, and F, respectively. Denote the angles of $\triangle ABC$ by α, β and γ, respectively. Because $\overline{AB}$ and

$\overline{AC}$ are both tangent to the incircle of $\triangle ABC$ at D and F, respectively, it follows that $AD = AF$ and thus $\angle ADF = \angle AFD = 90^\circ - \frac{\alpha}{2}$. Similarly, $\angle BDE = 90^\circ - \frac{\beta}{2}$, and thus

$$\angle EDF = 180^\circ - (90^\circ - \frac{\alpha}{2}) - (90^\circ - \frac{\beta}{2}) = \frac{\alpha+\beta}{2}.$$

Similar argument shows that the other angles of $\triangle DEF$ are $\frac{\alpha+\gamma}{2}$ and $\frac{\beta+\gamma}{2}$. If $\triangle ABC$ is similar to $\triangle DEF$, then the angles α, β, γ must be equal to the angles $\frac{\alpha+\beta}{2}, \frac{\alpha+\gamma}{2}, \frac{\beta+\gamma}{2}$ in some order. We can assume that $\alpha \leq \beta \leq \gamma$, which then implies $\frac{\alpha+\beta}{2} \leq \frac{\alpha+\gamma}{2} \leq \frac{\beta+\gamma}{2}$. But then we must have $\alpha = \frac{\alpha+\beta}{2}$, $\beta = \frac{\alpha+\gamma}{2}$ and $\gamma = \frac{\beta+\gamma}{2}$. Rearranging the first equation gives $\alpha = \beta$ and the third equation gives $\beta = \gamma$. This shows that the original triangle (and also $\triangle DEF$) must be equilateral.



4. Show that if x, y, z are positive numbers, then

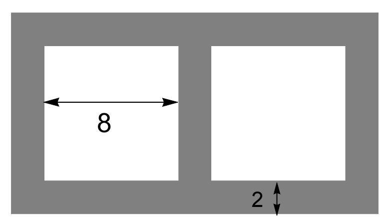
$$x^{2014}(x-y)(x-z) + y^{2014}(y-x)(y-z) + z^{2014}(z-x)(z-y) \geq 0.$$

SOLUTION. Note that the three terms on the right are symmetric: if we switch any two numbers within x, y, z, then the sum will stay the same. Thus, without loss of generality, we can assume that $0 < x \leq y \leq z$.

If $x = y$, the expression reduces to $z^{2014}(z-x)^2$ which is clearly positive. Similarly, if $y = z$, the expression is positive. So we can assume that $x < y < z$. Then the term $x^{2014}(x-y)(x-z)$ is positive. We will show that the sum of the other two terms is positive as well. From $z > y$ and $z - x > y - x$ it follows that

$$\begin{aligned} y^{2014}(y-x)(y-z) + z^{2014}(z-x)(z-y) &> -y^{2014}(y-x)(z-y) + z^{2014}(y-x)(z-y) \\ &= (z^{2014} - y^{2014})(y-x)(z-x) > 0. \end{aligned}$$

5. The diagram shows a 12×22 rectangular region with two 8×8 squares removed so that there is a border of width 2 around and between the squares. We would like to cut up the shaded region into non-overlapping triangles. Show that this is possible with 10 triangles, but impossible with 9 triangles.



SOLUTION. We can easily cut the region into 5 rectangles and then each of them could be cut further into 2 triangles. (See the first diagram below for an example.) We will show that it is impossible to cut the region into 9 triangles.

Consider the 10 unit length line segments in the inside and outside boundary colored red in the second diagram below. The midpoints of the vertical line segments are located 6 units from the top and the midpoints of the horizontal ones are located 6 units from the left side of the bounding big rectangle. When we cut the region into triangles, each of the marked line segments must intersect with a positive portion of a side of one of the triangles. (I.e. the intersection is more than just one point.) We will show that no triangle can intersect more than one of these 10 line segments, which will show that we have to have at least 10 triangles.

Suppose that we cut the region into triangles and one of those triangles intersect two of the marked line segments with its sides. Clearly, those two line segments cannot be parallel, since a triangle cannot have parallel sides. So the triangle must intersect a horizontal and a vertical line segment. But then that triangle must contain a line segment which connects that horizontal and vertical line (because the triangle is convex), and any such line would intersect the square holes in the region. This is impossible, since the triangles must all be within the region. The found contradiction shows that we cannot cut the region into fewer than 10 non-overlapping triangles.