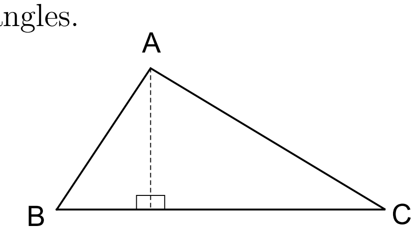
WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET I (2014-2015)

1. Are there functions $f(x)$ and $g(y)$ defined on all the real numbers so that $f(x)g(y) = x+y+1$ for every choice of x and y?

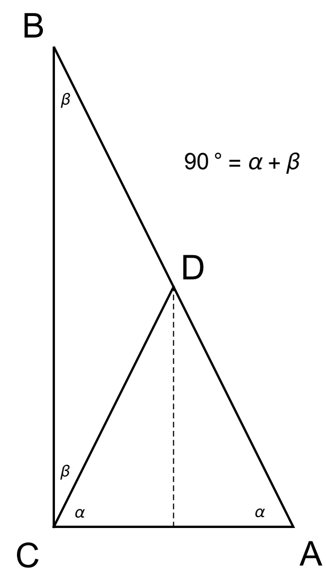
SOLUTION. We will show that there are no such functions. Suppose the opposite. Then with $x = y = -1/2$ we would get $f(-1/2)g(-1/2) = 0$. Thus $f(-1/2)$ or $g(-1/2)$ must be equal to 0. But if $f(-1/2) = 0$ then $x = -1/2$, $y = 1/2$ would give $f(-1/2)g(1/2) = 0 \neq -1/2 + 1/2 + 1 = 1$. We get into a similar contradiction when $g(-1/2) = 0$.

2. Prove that every triangle can be divided into 4 isosceles triangles.



SOLUTION. We first show that any triangle can be divided into two right triangles, and then that any right triangle can be divided into two isosceles triangles.

For the first part note that any triangle will have at least two acute angles. If B and C are acute angles in $\triangle ABC$, then dropping an altitude from the A will divide the triangle into two right triangles.



For the second part assume that $\triangle ABC$ has a right angle at C. Let the perpendicular bisector of $\overline{AC}$ intersect $\overline{AB}$ at point D. Then $\overline{AD} = \overline{CD}$ and this means that $\angle CAD = \angle ACD$ and $\triangle CAD$ is isosceles. Since $\angle ACB = 90^o$, we have $\angle DCB = 90^o - \angle ACD$. Because the angles of the right triangle $\triangle ABC$ sum to 180^o, we get $\angle DBC = \angle ABC = 90^o - \angle BAC = 90^o - \angle CAD = 90^o - \angle ACD$. Thus $\angle DBC = \angle DBC$ which means that $\triangle DCB$ is also isosceles. (Thus D is actually the midpoint of $\overline{AB}$.) This shows that every right triangle can be cut into two isosceles triangles, which finishes the proof.

3. We have a 5 by 5 grid of squares where the four corner squares are colored white and the rest of the squares are colored black. We can change the colors of any two adjacent squares (i.e. two squares that share a side) by flipping the colors of each of the two squares from black to white or from white to black. We can do this as many times as we wish. Is it possible to make all the squares in the grid white?

SOLUTION. It is not possible to turn the entire 5 by 5 grid of squares white. Each time colors are flipped, either 2 black squares become 2 white squares, 2 white squares become 2 black squares, or 1 white square and 1 black square switch colors. In each case the number of white squares either does not change or changes by 2. Since the number of white squares at the beginning is 4, there must always be an even number of white squares regardless of how many or which squares were flipped. Since we have an odd number of squares in the grid, this means that we cannot make them all white.

4. You have 2014 marbles each of which you can paint red, green, blue or yellow. Find the largest integer m so that no matter how you paint the marbles and no matter how you put them into 25 bags, at least one bag has at least m marbles of the same color.

SOLUTION. The answer is $m = 21$. Imagine that we subdivide each bag into 4 compartments according to our colors. Then we have 100 compartments, and so by the pigeonhole principle, one compartment has at least 21 marbles. (Since $20 \cdot 100 < 2014$.) Now we have to show that $m = 22$ does not work, i.e. that we can color the marbles and then arrange them in the bags so that each will have at most 21 of the same color. In order to do this imagine that we have 2100 marbles and we colored exactly one quarter of them with each of the four colors. Then we can put 21 colors of each color into each of the 25 bags. Removing any of the 86 marbles will now give a suitable arrangement.

5. Show that the minimal value of $\left|\frac{a}{b} - \frac{123}{2014}\right|$ is $\frac{1}{1883 \cdot 2014}$ if a, b are positive integers with $b < 2014$.

SOLUTION. It is easy to check that $\frac{1}{1883 \cdot 2014} = \frac{115}{1883} - \frac{123}{2014}$, so $\frac{1}{1883 \cdot 2014}$ can be achieved as a value for $\left|\frac{a}{b} - \frac{123}{2014}\right|$. Now we just have to show that we cannot go below that.

We may assume that a and b are relatively prime (otherwise we could simplify $\frac{a}{b}$). Since 123 and 2014 are also relatively prime, and $b < 2014$, $\frac{a}{b}$ cannot be equal to $\frac{123}{2014}$, and the minimal value of $\left|\frac{a}{b} - \frac{123}{2014}\right|$ must be positive.

We have

$$\left|\frac{a}{b} - \frac{123}{2014}\right| = \frac{|123b - 2014a|}{2014 \cdot b}.$$

If $|123b - 2014a| \geq 2$ then

$$\left|\frac{a}{b} - \frac{123}{2014}\right| = \frac{|123b - 2014a|}{2014 \cdot b} \geq \frac{2}{2013 \cdot 2014} = \frac{1}{1012 \cdot 2013} > \frac{1}{1883 \cdot 2014}.$$

Thus we just have to check the pairs a, b for which $123b - 2014a = 1$ or $123b - 2014a = -1$.

Plugging in $a = 115$, $b = 1883$ we get $123b - 2014a = -1$. We first show that there is no other positive integer solution of $123b - 2014a = -1$ with $b < 2014$. Suppose that there is another solution $123b' - 2014a' = -1$. Then subtracting these equations we would get $123(b - b') = 2014(a' - a)$. Thus $123(b - b')$ is divisible by 2014, and since 123 is relatively prime to 2014, we get that 2014 divides $b - b'$. But if $b \neq b'$ and 2014 divides $b - b' = 1883 - b'$, then $|1883 - b'| \geq 2014$ and b' could not be between 0 and 2013.

Now consider the equation $123b' - 2014a' = 1$ with $0 < b' < 2014$. If there is a solution, then adding it to the equation $123b - 2014a = -1$ with $a = 115, b = 1883$ we would get $123(b + b') = 2013(a + a')$. The same argument as before gives that 2014 divides $b + b' = 1883 + b'$. Since $0 < 1883 + b' < 2014 \cdot 2$, this would imply that $1883 + b' = 2014$ and $b' = 131$. From this we get $a' = \frac{123 \cdot 131 - 1}{2014} = 8$. But for this pair of numbers

$$\left|\frac{a'}{b'} - \frac{123}{2014}\right| = \frac{|123b' - 2014a'|}{2014 \cdot b'} = \frac{1}{2014 \cdot 8} > \frac{1}{1883 \cdot 2014}.$$

We checked all possibilities, and showed that $\left|\frac{a}{b} - \frac{123}{2014}\right|$ is always at least $\frac{1}{1883 \cdot 2014}$.