WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

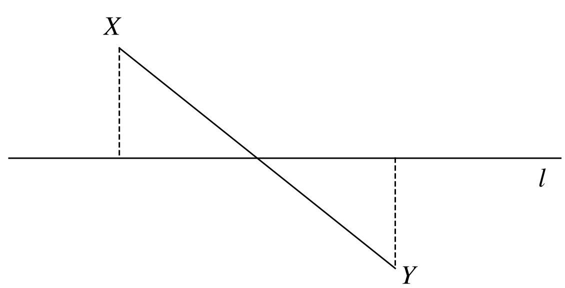
SOLUTIONS TO PROBLEM SET V (2013-2014)

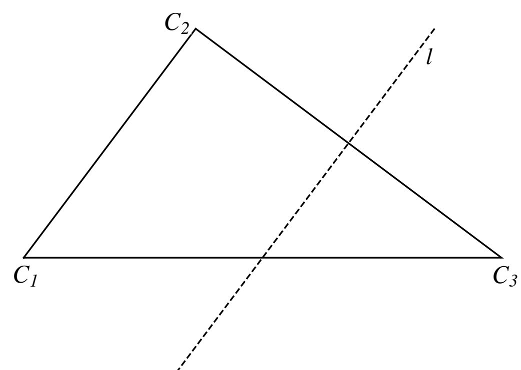
1. In the coordinate plane, we have points $C_1 = (0, 0)$, $C_2 = (18, 24)$, and $C_3 = (50, 0)$. Points A and B are in the plane so that the triangles $\triangle ABC_1$, $\triangle ABC_2$, and $\triangle ABC_3$ all have area 1200. Find all the possible lengths for the segment $\overline{AB}$.

SOLUTION. We start by noting that the area of the triangle $\triangle C_1C_2C_3$ is $\frac{50\cdot 24}{2} = 600$, since the length of $\overline{C_1C_3}$ is 50 and the corresponding altitude is 24.

Suppose $\overline{AB}$ is a line segment such that $\triangle ABC_1$, $\triangle ABC_2$, and $\triangle ABC_3$ all have area 1200, and let ℓ be the line that passes through A and B. The area of each $\triangle ABC_i$ (with $i = 1, 2, 3$) is half the length of the base $\overline{AB}$ times the altitude of the triangle to that base. The altitude is the distance from C_i to ℓ, so, if all three areas are equal to 1200, then C_1, C_2 and C_3 are the same distance from ℓ.

Suppose that two points X and Y are the same positive distance from a line ℓ. If the two points are on the same side of ℓ then that means that the line passing through them is parallel to ℓ. If the two points are on different sides of ℓ then ℓ must pass through the midpoint of the line segment XY. This can be seen by drawing the lines through X and through Y that are perpendicular to ℓ and noting that the produced two right-angled triangles are congruent.





Returning to our problem, because C_1, C_2, and C_3 are not on the same line, it cannot be that all three points are on the same side of ℓ. Thus, two of the points must lie on one side of ℓ, and the third point must lie on the opposite side. If C_i and C_j are on one side of ℓ and C_k is on the other side, then the line C_iC_j is parallel to ℓ, and ℓ must pass through the midpoints of the line segments C_iC_k and C_jC_k. Then the distance of these three points from ℓ is exactly half the distance from C_k to the line C_iC_j, which is the appropriate altitude in $\triangle C_1C_2C_3$. We can do this three different ways as there are three choices for C_k.

We want the areas of $\triangle ABC_1$, $\triangle ABC_2$, and $\triangle ABC_3$ to be twice as big as the area of $\triangle C_1C_2C_3$, the length of $\overline{AB}$ must be four times as big as the side $\overline{C_iC_j}$. (Since the length of the altitude corresponding to AB in the three triangles are half the length of the altitude corresponding to C_iC_j in $\triangle C_1C_2C_3$.) We note that taking any segment AB along the ℓ determined above with this length will satisfy the conditions of the problem. The sides of $\triangle C_1C_2C_3$ have lengths 50, $\sqrt{24^2 + 18^2} = 30$ and $\sqrt{24^2 + (50 - 18)^2} = 40$. Thus the length of $\overline{AB}$ can take the following three values: 200, 120 and 160.

2. Find all functions f from the positive real numbers to the real numbers such that for all positive real numbers x and y we have $f(x+y) = f(xy)$.

SOLUTION. As long as $s^2 > 4p > 0$, there are positive numbers x, y with sum s and product p. We can see this because in that case the polynomial $t^2 - st + p$ has two real roots x and y with $x + y = s$ and $xy = p$. Since their product is positive they have the same sign, and since their sum is positive they must be both positive.

So taking $p = 1/4$ and any $s > 1$ there are positive x, y with $x + y = s$ and $xy = 1/4$ which means that for all such s we have $f(s) = f(1/4)$. Similarly, taking $s = 20$ and any $0 < p < 100$, we have $f(p) = f(20) = f(1/4)$. This means that the function f has to be a constant on the positive numbers, and of course any such function satisfies the identity.

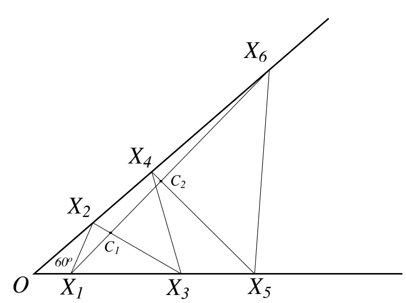
3. We have a 2014×2014 board made up of 2014^2 unit squares. We would like to cover the board in a single layer with 3×3 and 4×4 tiles (made up of 9 and 16 unit squares, respectively). Each used 3×3 (or 4×4) tile will have to cover exactly 9 (or 16) of the board's unit squares, and each unit square will have to be covered by exactly one of the used tiles. Is it possible to find such a tiling?

SOLUTION. It is not possible to construct such a tiling. Let us color the columns of the 2014×2014 board with red, blue and green, periodically: first column red, second blue, third green and then repeating the pattern. Then there will be 2014 more red squares than blue squares (because the last column is also red). Whenever we place a 3×3 tile on the board we will cover the same number of red and blue squares. When we place a 4×4 square then the difference between the covered red and blue squares can be 4, 0 or -4, which is always a multiple of 4. Thus if we could cover all the squares of the board with 3×3 and 4×4 squares then the difference between the covered red and blue tiles would be a multiple of 4. But 2014 (the difference) is not a multiple of 4, which shows that the tiling is impossible.

4. Let $X_1, X_2, \ldots, X_6$ be the vertices of a hexagon. Suppose that O is a point in the hexagon such that for all $1 \le i \le 6$ we have $\angle X_i O X_{i+1} = 60^\circ$ (where $X_7 = X_1$). Suppose $OX_1 < OX_3 < OX_5$ and $OX_2 < OX_4 < OX_6$. Show that

$$X_1X_2 + X_3X_4 + X_5X_6 < X_2X_3 + X_4X_5 + X_6X_1.$$

SOLUTION. The lengths X_iX_{i+1} only depend on the lengths OX_i and OX_{i+1} and the angle $60^\circ = \angle X_iOX_{i+1}$. Suppose that we place X_1, X_3, X_5 along a ray starting at O so that the distances from O are the same as before, and then place X_2, X_4, X_6 along a ray starting at O that is 60° away. Then the distances X_iX_{i+1} will be the same as before. Let C_1 be the intersection of $\overline{X_2X_3}$ and $\overline{X_6X_1}$ and C_2 be the intersection of $\overline{X_4X_5}$, and $\overline{X_6X_1}$. By the triangle inequality, we have $X_1C_1 + C_1X_2 > X_1X_2$, and $X_3C_1 + C_1C_2 + C_2X_4 > X_3X_4$. Also $X_5C_2 + C_2X_6 > X_5X_6$. We add up these three inequalities. By the configuration of the X_i, we have that C_1 is between X_2 and X_3, C_2 is between X_4 and X_5 and that X_1, C_1, C_2, X_6 lie along a line in that order. Thus the inequalities add to $X_2X_3 + X_4X_5 + X_6X_1 > X_1X_2 + X_3X_4 + X_5X_6$.



5. Is it possible to divide the numbers $1, 2, \ldots, 1000$ into two groups so that the sum of the squares of one group is equal to the sum of the squares of the other?

SOLUTION. We will show that the answer is yes. First, we prove that for any 8 consecutive integers we can divide them into two groups of fours so that the sum of the squares are the same. Indeed, if we denote the 8 integers by $n-3, n-2, \ldots, n+3, n+4$ then

$$(n-3)^2 + n^2 + (n+2)^2 + (n+3)^2 = (n-2)^2 + (n-1)^2 + (n+1)^2 + (n+4)^2.$$

This can be readily justified by expanding the parentheses and checking that both sides are equal to $4n^2 + 4n + 22$. Since 1000 is a multiple of 8, we can first divide the first 1000 numbers into 125 groups of consecutive 8 numbers, and then we can use the previous construction to divide each group of 8 into two groups where the sum of the squares agree. Picking one of these groups of 4 for each group of 8 will give the appropriate construction. E.g. these two groups will work:

$$\{1+8k, 4+8k, 6+8k, 7+8k : 0 \leq k \leq 124\}, \text{ and } \{2+8k, 3+8k, 5+8k, 8+8k : 0 \leq k \leq 124\}.$$