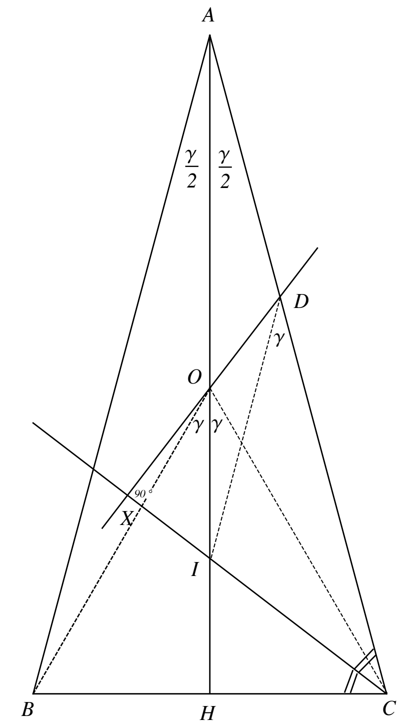
WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET IV (2013-2014)

1. Let ABC be a triangle with $AB = AC$, circumcenter O and incenter I. Let D be the point on AC such that line OD is perpendicular to line CI. Prove that ID and AB are parallel.

SOLUTION. Let H be the foot the altitude from A to BC (which goes through O and I by symmetry since $AB = AC$). If we let OD and CI intersect at X, then DXC and IHC are both right triangles, and $\angle DCX = \angle ICH$ since the incenter is on the angle bisector at C. So the remaining angles are equal: $\angle HIC = \angle XDC = \angle ODC$. Thus $\angle OIC = 180 - \angle HIC = 180 - \angle ODC$, and $ODCI$ are on a circle. Thus, $\angle IDC = \angle IOC = \angle HOC$. We have that $\angle HOC = \angle BOC/2$ by symmetry, and $\angle BAC = \angle BOC/2$ by the fact that O is the center of circle ABC. So, $\angle IDC = \angle BAC$, which proves AB and ID are parallel.



2. Show that if the sum of the fifth powers of five integers is divisible by 25 then one of the original integers is divisible by five.

SOLUTION. Suppose that there are five numbers not divisible by five so that the sum of the fifth powers is divisible by 25. Then each of these numbers can be written in the form of $5k+1, 5k-1, 5k+2$ or $5k-2$ with an integer k. We have

$$\begin{aligned}(5k+a)^5 &= a^5 + 25a^4k + 250a^3k^2 + 1250a^2k^3 + 3125ak^4 + 3125k^5 \\ &= a^5 + 25(a^4k + 10a^3k^2 + 50a^2k^3 + 125ak^4 + 125k^5).\end{aligned}$$

This means that the fifth power of an integer has the same residue modulo 25 as the fifth power of its residue modulo 5. Checking the residues of $1^5, (-1)^5, 2^5, (-2)^5$ modulo 25 we see that the fifth powers of our integers modulo 25 can only be $1, -1, 7, -7$. Replacing the fifth powers with their residues we would get five numbers with each equal to $1, -1, 7$ or -7 so that the sum is divisible by 25. Since the sum of five such numbers is between -35 and 35, the sum could only be $-25, 0$ or 25. ± 1 and ± 7 are all odd, so adding five of them would also give an odd number. This means that the sum must be 25 or -25. But it is easy to check that this is impossible. E.g. to get 25 we would need at least four sevens, but we cannot find a fifth number from the allowed set to make the sum equal to 25. The contradiction shows that if the sum of the fifth powers is divisible by 25 then one of the numbers is divisible by 5.

3. We have an urn with 1000 balls numbered from 1 to 1000. We choose 9 balls randomly from the urn (without replacement) and add the shown numbers. Determine the probability that the sum is even.

SOLUTION. We will show that if we choose 9 balls randomly then it is equally likely to get an even sum as an odd sum, i.e. the probability in question is $\frac{1}{2}$. View the 1000 balls in the urn as 500 even-numbered balls painted white and 500 odd-numbered balls painted black. The sum of the number on the 9 chosen balls will be odd exactly when we select an odd number of white balls and an even number of black balls. Thus, we could select 1, 3, 5, 7, or 9 white balls to get an odd sum or 0, 2, 4, 6, or 8 white balls to get an even sum. Clearly, since we begin with 500 balls of each color, the probability of selecting k white balls with $9-k$ black balls is equal to the probability of selecting k black balls with $9-k$ white balls. This shows that the probabilities of selecting 1, 3, 5, 7, or 9 white balls is equal to the probability of selecting 8, 6, 4, 2, or 0 white balls, respectively. Therefore, it is equally likely that our sum will be even or odd.

4. Ann and Beth play the following game. They start out with n dimes on a table and take turns with Ann starting. In each step a player can take at most $n/2+1$ dimes from the table, but she has to take at least one. If somebody takes all the dimes on the table then she wins. For which values of n will Ann have a winning strategy?

SOLUTION. We will call a number $n \geq 1$ a *winning number* if Ann has a winning strategy and a *losing number* otherwise. Clearly, 1 and 2 are winning numbers, since Ann can take all the dimes in one step. That makes 3 a losing number: whatever move Ann makes, Beth will have 1 or 2 dimes on the table and she can take the remaining dimes. The numbers 4, 5, 6, 7 and 8 are all winning numbers again: Ann can always take away a certain number of dimes so that there are 3 left, and then Beth cannot win. This means that 9 is losing number: Ann cannot get to 3 in a single step, and whatever she does, Beth can get to 3 in a single step.

We can generalize the observed pattern: if we already know all the losing numbers below n then Ann can win starting from n if she can get to one of them in a single step, otherwise she will lose. If k is a losing number, then the next losing number is $2k+3$, as we can get to k from $k+1, k+2, \dots, 2k+2$ in a single (allowed) step, but not from $2k+3$. This means that the losing numbers are given by the sequence $a_k, k \geq 1$, where $a_1 = 3$ and $a_{k+1} = 2a_k + 3$ for $k \geq 1$. Note, that then we have $a_{k+1} + 3 = 2(a_k + 3)$ so

$$a_{k+1} + 3 = 2(a_k + 3) = 4(a_{k-1} + 3) = \dots = 2^k(a_1 + 3) = 6 \cdot 2^k,$$

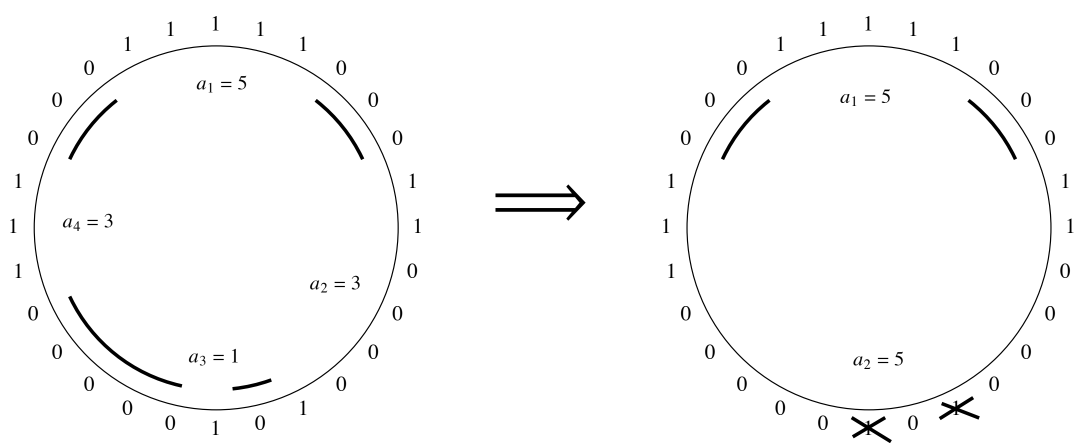
and $a_{k+1} = 6 \cdot 2^k - 3 = 3(2^{k+1} - 1)$. Thus all the numbers of the form $3(2^{k+1} - 1)$ (with nonnegative integer k) are losing numbers, and all the others are winning numbers.

5. The numbers from 1 to 100 are written in order around a circle. On each move Alice chooses an even number, y, on the circle, erases it along with its two neighbors, x and z, and replaces the three numbers with the sum of the two neighbors, $x+z$. She continues to make moves until either all the remaining numbers are odd or there are fewer than 3 numbers remaining. Prove that no matter how she chooses her moves, she will end up with 2 even numbers.

SOLUTION. We can replace each number with its residue modulo 2. Then each move has one of the following forms: $000 \to 0, 001 \to 1, 100 \to 1, 101 \to 0$. In other words: in each step we can delete two zeros next to each other, or two ones if they are the neighbors of a zero. In particular, the number of zeros around the circle will always be even, and the number of ones around the circle will always be even. (Since we start with 50 for both.)

As we go around the circle we see blocks of zeros and blocks of ones alternating. At the beginning, these blocks are of size 1, but later on they can be bigger. We will be interested in the blocks of zeros which have an *odd* length. (Initially all of them are odd.) Suppose that after some steps we have k such blocks. Note, that k must be even, since otherwise we would have an odd number of zeros all together. We count the ones between the neighboring odd blocks of zeros and denote these numbers by $a_1, a_2, \dots, a_k$. (It does not matter, where we start the labeling.) If we don't have any blocks of zeros with odd length, then we set $k = 1$, and a_1 is just the total number of ones. At the beginning $k = 50$ and $a_i = 1$ for all i. (See the picture for another example.)

Now let's see what happens to these a_i values if we change the numbers in a step. If we delete two neighboring zeros then nothing will change. (We may have changed the number of zeros in an odd zero block, but we couldn't delete the whole block.) The change $101 \to 0$ will reduce two neighboring a_i values by 1, possibly eliminating one or both of those a_i terms. Note that if two adjacent a_i values are reduced to two positive values, then the sum $|a_1 - a_2 + \dots - a_k|$ will not change. If two adjacent a_i values are both reduced from 1 to 0 and, thus, are eliminated, the sum $|a_1 - a_2 + \dots - a_k|$ will also remain unchanged. If only one a_i is reduced to zero while its neighbor is reduced by 1 to a positive number, it means that two adjacent odd lengthened blocks of zeros have been merged resulting in one block of even length. For example, if this happens to a_{i-1}, $a_i = 1$, a_{i+1}, the three adjacent terms will be replaced by the one term $a_{i-1} + a_{i+1} - 1$. Here again the sum $|a_1 - a_2 + \dots - a_k|$ remains unchanged. In all cases, the sum $|a_1 - a_2 + \dots - a_k|$ will remain unchanged. At the start, this quantity is 0. If at any time there are no even numbers left, but $m > 0$ odd numbers, then this quantity would be $m > 0$. This cannot happen, so Alice must end up with 2 even numbers.



An example for a step when two of the a_i values disappear. The bold arcs show the odd zero blocks in the two configurations.