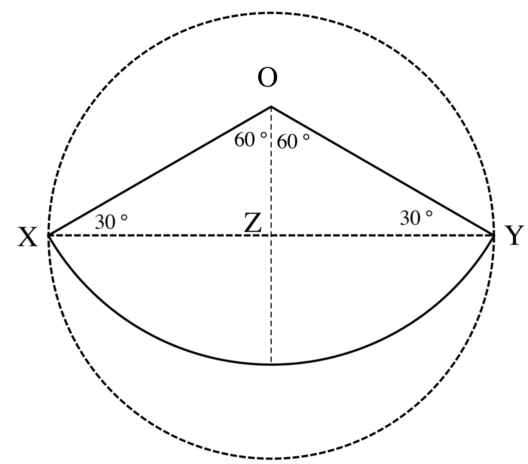
WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET III (2013-2014)

1. Somebody chose three points inside a circle of radius 1. Show that we can always choose two of the three points so that they are no more than distance 7/4 apart. (Make sure to give complete justifications in your argument.)

SOLUTION. Let the points be labeled A, B, C and the center of the circle O. The rays OA, OB, OC divide the disk into three circular sectors. Since the angles of these sectors add up to $360°$, one of these is at most $120°$. That means that we can always choose a circular sector with angle $120°$ and radius one that contains at least two of the three points. We will show that such a circular sector can be covered by a disk of diameter $\sqrt{3}$, this will prove that the distance of the two points in the sector is at most $\sqrt{3} < 7/4$.

Suppose that OXY is a circular sector with angle $120°$, radius one and O as the center. Let Z be the midpoint of XY, we will show that the sector is covered by a disk with center Z and diameter XY. The triangles $\triangle XZO$ and $\triangle OZY$ are both $30° - 60° - 90°$ triangles with a hypothenuse equal to 1. Thus $OZ = 1/2$, $XZ = ZY = \sqrt{3}/2$ and $XY = \sqrt{3}$. If we draw the circle with center Z and radius $XZ = \sqrt{3}/2$, then this covers the triangle XOY (since $OZ = 1/2 < \sqrt{3}/2$) and also the arc $\widehat{XY}$ (since the height of this is $1 - OZ = 1/2 < \sqrt{3}/2$. This shows that the circle of radius $\sqrt{3}/2$ covers the whole circular sector, which means that any two point inside the sector will be of distance at most $\sqrt{3}$.



2. Find all ordered triples (a, b, c) that satisfy the following system of equations:

$$ab + c = 6, \qquad bc + a = 6, \qquad ca + b = 6.$$

SOLUTION. Multiply the first equation by c, the second equation by a, and subtract the results to get $(abc + c^2) - (abc + a^2) = 6c - 6a$ which reduces to $(c - a)(c + a) = 6(c - a)$. Thus, either $a = c$ or $a + c = 6$.

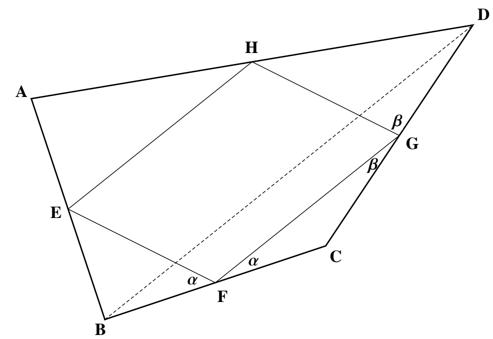
If $a = c$ then the third equation gives $b = 6 - a^2$. Substituting into the first equation gives $a(6 - a^2) + a = 6$ which reduces to $0 = a^3 - 7a + 6 = (a + 3)(a - 2)(a - 1) = 0$. It follows that $a = -3$, $a = 2$, or $a = 1$ with $c = a$ and $b = 6 - a^2$. This gives the following solutions for (a, b, c): $(-3, -3, -3), (2, 2, 2), (1, 5, 1)$.

If $a + c = 6$ then the third equation gives $b = 6 - a(6 - a)$ and plugging this into the first one yields $a(6 - a(6 - a)) + (6 - a) = 6$. We can reduce this to $0 = a^3 - 6a^2 + 5a = a(a - 1)(a - 5)$. Thus a can be 0, 1, or 5 and $c = 6 - a$, $b = 6 - ac$. The $a = 0$ case doesn't give a good solution, while the other two cases give $(1, 1, 5)$ and $(5, 1, 1)$.

Thus, there are five ordered triples that satisfy the system of equations.

3. We have a special billiards table in the shape of a convex quadrilateral $ABCD$. A billiard ball starts at the midpoint of side AB and is hit so that it bounces off side BC at its midpoint, side CD at its midpoint, side DA at its midpoint, and returns to the position on side AB where it started. Find all shapes of quadrilateral $ABCD$ that would allow this to happen. (When the ball bounces off of one of the sides, the angle measured between the incoming path and the side of the table is always the same as the angle measured between the outgoing path and the side of the table.)

SOLUTION. We will show that the quadrilateral must be a rectangle. Let E, F, G, and H be the midpoints of sides AB, BC, CD, and DA, respectively. Note that EH is the midline of $\triangle ABD$ so it is parallel to BD. (This can be seen by comparing the similar triangles $\triangle ABD$ and $\triangle AEH$.) The same is true for FG and diagonal BD which means that EH and FG are parallel. Similarly, EF is parallel to GH, and quadrilateral $EFGH$ is a parallelogram.



When a ball bounces off the side of the table, its angle of incidence is equal to its angle of reflection, that is, $\angle BFE = \angle CFG = \alpha$ and $\angle CGF = \angle DGH = \beta$ for some α and β. If we add all the angles at F and G then we get

$$2 \cdot 180° = 360° = 2\alpha + 2\beta + \angle EFG + \angle FGH.$$

Since $EFGH$ is a parallelogram, we have $\angle EFG + \angle FGH = 180°$ and this gives $\alpha + \beta = 90°$. By looking at the angles in $\triangle FCG$ we conclude that $\angle FCG = \angle BCD = 90°$. The same argument shows that the other angles of $ABCD$ are also equal to $90°$ which means that it must be a rectangle. Reversing our steps shows that if $ABCD$ is a rectangle then $EFGH$ can be the path of the billiard ball.

4. We have an urn with 3 red and k white balls. Let p be the probability that if we pick two balls randomly with replacement (i.e. we replace the first ball before choosing the second), then both balls are red. Let q be the probability that if we pick three balls randomly without replacement (i.e. we keep choosing from the remaining balls in each step), then all three balls are red. Find the value of k if we know that $p = 5q$.

SOLUTION. Suppose that we numbered the balls from 1 to $3 + k$, with the balls 1, 2 and 3 being red. When we choose two balls with replacement then there are $(k + 3)^2$ different pairs we can choose (since there are $k + 3$ choices for the first one and $k + 3$ for the second) and $3^2 = 9$ of them will have two red balls. Since all the outcomes are equally likely, the probability of getting two red balls is $p = \frac{9}{(k+3)^2}$. When we choose three balls without replacement, then there are $(k + 3)(k + 2)(k + 1)$ different outcomes (since there are $k + 3$ choices for the first pick, but only $k + 2$ for the second, and just $k + 1$ for the third). Out of these outcomes $3 \cdot 2 \cdot 1 = 6$ will have three red balls (we have three choices for the first red, two for the second and one for the last pick). This gives $q = \frac{6}{(k+3)(k+2)(k+1)}$. We know that $p = 5q$ so we have $\frac{9}{(k+3)^2} = 5\frac{6}{(k+3)(k+2)(k+1)}$. Cross-multiplying and rearranging the terms gives

$$0 = 9(k + 2)(k + 1) - 30(k + 3) = 9k^2 - 3k - 72.$$

Solving this equation we get $k = -8/3$ (which is not a possible solution) and $k = 3$. Thus the answer is $k = 3$ and $p = \frac{9}{36} = \frac{1}{4}$, $q = \frac{6}{6 \cdot 5 \cdot 4} = \frac{1}{20}$.

5. Consider the number $n = 2014^{2014^{2014}}$. What is the first decimal digit after the decimal point in the number $\sqrt{n^2+n+1}$?

SOLUTION. If we evaluate $\sqrt{n^2+n+1}$ for small values of n, we find that the first digit after the decimal point is 5 if $n > 3$. We will show that this is indeed the case. The integer part of $\sqrt{n^2+n+1}$ is n since $n < \sqrt{n^2+n+1} < n+1$. To show that the first digit point is 5, it is enough to prove that

$$n + 1/2 \leq \sqrt{n^2+n+1} < n + 3/5, \qquad \text{for } n > 3.$$

The lower bound follows from

$$(n+1/2)^2 = n^2 + n + 1/4 < n^2 + n + 1.$$

For the upper bound we need

$$(n+3/5)^2 = n^2 + 6/5n + 9/25 > n^2 + n + 1$$

which follows from $n/5 > 16/25 > 3$. This shows that the first digit after the decimal is 5 for any integer $n > 3$, in particular for $n = 2014^{2014^{2014}}$.