WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET II (2013-2014)

1. We wrote integer numbers on the vertices of a tetrahedron (one for each vertex). Show that if the sum of the numbers on each face is divisible by five then, all the numbers are divisible by five.

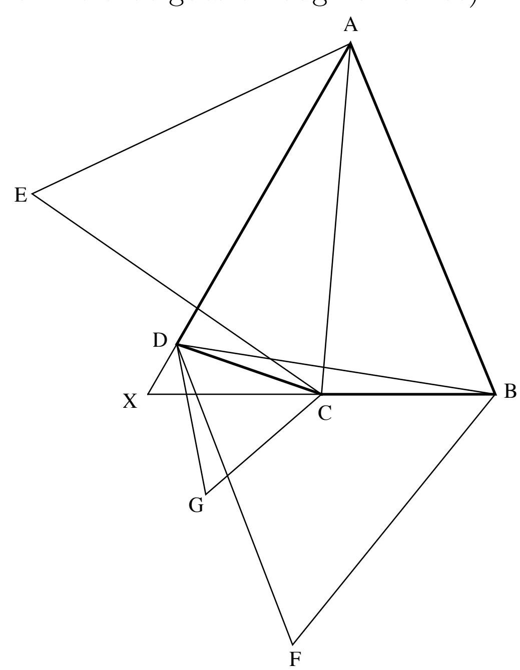
SOLUTION. Denote the numbers on the vertices by a_1, a_2, a_3 and a_4. We know that the following sums are all divisible by 5:

$$s_1 = a_2 + a_3 + a_4, \quad s_2 = a_1 + a_3 + a_4, \quad s_3 = a_1 + a_2 + a_4, \quad s_4 = a_1 + a_2 + a_3.$$

Then $s_1+s_2+s_3+s_4 = 3(a_1+a_2+a_3+a_4)$ is also divisible by five, which means $a_1+a_2+a_3+a_4$ is also a multiple of five. But then this is true for $a_1+a_2+a_3+a_4-s_1 = a_1$, and a similar argument shows that a_2, a_3 and a_4 are all divisible by 5 as well.

2. Let $ABCD$ be a quadrilateral such that $\angle DAB + \angle ABC = 120°$. Construct equilateral triangles ACE, BDF, CDG, with E, F, G each on the other side of AC, BD, CD (respectively) from AB. Show that E, F, G are collinear (i.e. there is a line that goes through all three).

SOLUTION. Let AD and BC intersect at X. Then $\angle CXD = 60°$ from $\angle DAB + \angle ABC = 120°$. Since $\angle CGD = 60° = \angle CXD$, we have that $CDGX$ are on a circle, and thus if G is on the same side of line DX as C and B, then $\angle DXG = 180° - \angle DCG = 120°$ and if G is on the other side of line DX as C and B, then $\angle DXG = \angle DCG = 60°$. Similarly, $ACXE$ are on a circle and $BDXF$ are on a circle. Thus, if F is on the same side of line DX as C and B, then $\angle DXF = 180° - \angle DBF = 120°$, and if F is on the other side of line DX as C and B, then $\angle DXF = \angle DBF = 60°$. In any of these cases, we see that FXG are collinear. An analogous argument shows that EXG are collinear. Thus, it follows that E, F, and G are collinear.



3. Let $n > 1$ be an integer and suppose that the polynomial $p(x)$ has degree n and satisfies $p(1) = 3$, $p(2) = 5$, $p(3) = 7$, $\ldots, p(n) = 2n+1$, and $p(n+1) = 2n+5$. Evaluate $p(n+3)$.

SOLUTION. Consider the polynomial $q(x) = p(x) - (2x+1)$. This is zero for the numbers $1, 2, 3, \ldots n$ so we can factor out $x - i$ for each $1 \le i \le n$ from $q(x)$. But $q(x)$ has degree n (since $p(x) = q(x) + 2x + 1$ has degree n) which means that there must be a constant c so that $q(x) = c(x-1)(x-2)(x-3)\cdots(x-n)$ and $q(n+1) = cn!$. But we also know that $p(n+1) = 2n+5$ so we also have $q(n+1) = 2n+5-(2(n+1)+1) = 2$. This gives $c = \frac{2}{n!}$ and

$$p(n+3) = q(n+3) + 2(n+3) + 1 = \frac{2}{n!}(n+2)(n+1)\cdots 3 + (2n+7)$$
$$= (n+2)(n+1) + (2n+7) = n^2 + 5n + 9.$$

4. Is there a function $f(x)$ defined on all real numbers x so that $f(f(x)) = -x$ for all x? (Either prove that there is no such f, or give an example of one.)

SOLUTION. The answer is yes and, we will construct an example of such a function. Set $f(0) = 0$. Then $f(f(x)) = -x$ for $x = 0$. For a given $x \neq 0$ if we start to apply f to the number repeatedly, then we will get back to x after four steps:

$$x \to f(x) \to -x \to f(-x) \to x.$$

Our strategy for the construction of f will be as follows: we first pair up the positive numbers in ordered pairs (a, b) so that each positive number appears exactly in one pair. Then we define f in a way that $f(a) = b$, $f(b) = -a$, $f(-a) = -b$ and $f(-b) = a$. This way f will be defined for all numbers, and $f(f(x)) = -x$ will hold for all nonzero x.

The pairing up of the positive numbers can be done in various different ways. One possibility is the following: if $x \in (2n, 2n+1]$ with a nonnegative integer n then we pair up x with $x+1$. This way each positive number will be covered in a pit exactly once. This leads to the following function:

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ x+1 & \text{if } 2n < x \leq 2n+1, n \text{ is a non-negative integer}, \\ -(x-1) & \text{if } 2n+1 < x \leq 2n+2, n \text{ is a non-negative integer}, \\ x-1 & \text{if } 2n < -x \leq 2n+1, n \text{ is a non-negative integer}, \\ -(x+1) & \text{if } 2n+1 < -x \leq 2n+2, n \text{ is a non-negative integer}. \end{cases}$$

5. 2013 students each roll 9 standard six-sided dice and record how many times each of the numbers 1, 2, 3, 4, 5, and 6 appears. Show that there are at least two students who record the same result.

SOLUTION. Suppose a student rolls 9 dice, and for each integer k with $1 \leq k \leq 6$ the student sees the number k appearing on a die a_k times. Then $a_1 + a_2 + \cdots + a_6 = 9$. Denote the number of different solutions to this equation where each a_k is a nonnegative integer by N. If we can show that $N < 2013$, then by the Pigeonhole Principle there will be at least two students who wrote down the same six numbers $(a_1, a_2, \ldots, a_6)$.

To compute N, we encode a solution $(a_1, a_2, \ldots, a_6)$ as a sequence of 9 zeros and 5 ones the following way. We first write down a_1 zeros (none if $a_1 = 0$), and then a one, then a_2 zeros, and again a one, and repeat it with a_3, a_4, a_5, and finally finish the sequence with a_6 zeros. This way we used up nine zeroes and five ones, e.g. the solution $(1, 2, 3, 2, 0, 1)$ would be encoded as $0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 1, 1, 0$. Moreover, the sequences containing nine zeros and five ones can be exactly matched up with the solutions $(a_1, a_2, \ldots, a_6)$ so their number is exactly N. But these sequences are easy to count: we just have to choose the positions of the five ones out of the possible $9 + 5 = 14$ places, and this can be done in $\binom{14}{5} = \frac{14 \cdot 13 \cdot 12 \cdot 11 \cdot 10}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 2002$ ways. Thus, $N = 2002$, and since this is less than 2013, we indeed must have at least two students who wrote down the same six numbers.