WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

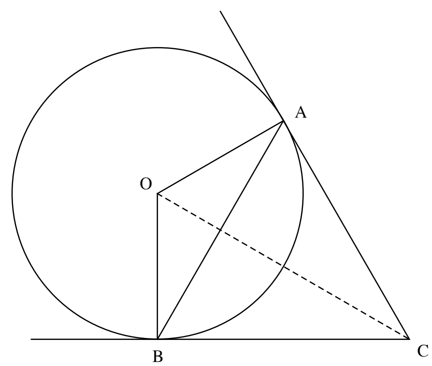
SOLUTIONS TO PROBLEM SET I (2013-2014)

1. What is the smallest positive integer n such that a cube of side length n can be cut into 2013 smaller cubes with integer side lengths?

SOLUTION. The answer is $n = 13$. Each smaller cube has volume at least 1, so we must have $n^3 \geq 2013 > 12^3$. We note that $6(3^3) + 4(2^3) + 2003(1^3) = 2197 = 13^3$. Inside of a $13 \times 13 \times 13$ cube, we can, in the bottom $3 \times 13 \times 13$ place 6 cubes of side length 3 and 4 of side length 2 and fill the rest with cubes of side length 1.

2. Equilateral triangle ABC has sides AC and BC tangent to a circle with center O at A and B, respectively. If $AO = \sqrt{3}$ what is the area of the quadrilateral $AOBC$?

SOLUTION. Since AC is tangent to the circle centered at O at A, we have that $\angle OAC = 90°$ and similarly, $\angle OBC = 90°$. We have that triangles OAC and OBC are congruent by side-side-side. So $\angle ACO = \angle OCB = 30°$. Thus OAB and OBC are $30 - 60 - 90$ right triangles, and so $OC = 2\sqrt{3}$ and $AC = BC = 3$, and triangles OAC and ABC both have area $3\sqrt{3}/2$. Then $AOBC$ has area $3\sqrt{3}$.



3. We distributed 100 balls into 100 boxes, and we didn't put all the balls into a single box. (There may be empty boxes.) Show that there is an integer k with $1 \leq k < 100$ so that we can choose k boxes which, together, contain exactly k balls.

SOLUTION. If we have a box with a single ball then we are done: we can let $k = 1$ and choose that box. If each box contains at least two balls then we have at least 50 empty boxes. Since not all of the balls are in the same box, we must have a box with at most 50 balls. If this box contains $2 \leq k \leq 50$ balls then choosing another $k - 1$ empty boxes (which we can as $k - 1 \leq 50$) we will have exactly k boxes with k balls in them.

Note: some students received a version of the problem set that contained the incorrect condition $1 \leq k \leq 100$. This makes the problem trivial, as $k = 100$ always satisfies the conditions. We gave full credit for this observation if the contestant worked with the problem with the typo in it.

4. A palindrome is a number which reads the same forward and backwards such as 1441 or 35253. Find the largest five-digit palindrome that is divisible by 101.

SOLUTION. Suppose there are digits a, b, and c so that the palindrome $\overline{abcba}$ is a multiple of 101. Since 1010 is a also a multiple of 101, we conclude that $\overline{abcba} - b \cdot 1010 = \overline{a0c0a}$ is a multiple of 101. Since $10201 = 10100 + 101$ is a multiple of 101, conclude that $\overline{a0c0a} - a \cdot 10201 = (c - 2a)100$ is a multiple of 101. Since 100 and 101 are relatively primes, 101 must divide $c - 2a$. But $-18 \leq c - 2a \leq 9$, and the only integer multiple of 101 in that range is zero.Thus $c - 2a = 0$ which means that the maximal value for the first digit of our palindrome is $a = 4$ and then c is 4. We can choose b any way we like and its maximal value is 9, which gives 49894 as the largest five-digit palindrome that is divisible by 101.

5. We multiplied four consecutive integers and the result was the same as the product of two consecutive integers. What are the possible values of the product?

SOLUTION. We will show that the product must be zero. If we take the consecutive integer numbers 0, 1, 2, 3 and 0, 1, respectively, then the products are equal to zero.

Suppose now that there are integers x and y so that

$$x(x+1)(x+2)(x+3) = y(y+1)$$

and these products are not equal to zero. Then $y < -1$ or $y > 0$. Note that

$$\begin{aligned} x(x+3) = x^2 + 3x &= (x^2 + 3x + 1) - 1, \qquad \text{and} \\ (x+1)(x+2) = x^2 + 3x + 2 &= (x^2 + 3x + 1) + 1. \end{aligned}$$

Using $(A - B)(A + B) = A^2 - B^2$ we get

$$x(x+1)(x+2)(x+3) = ((x^2 + 3x + 1) - 1)((x^2 + 3x + 1) + 1) = (x^2 + 3x + 1)^2 - 1.$$

This means that $y(y+1)$ is one less than a square number, or in other words: $y(y+1)+1 = y^2+y+1$ is a square. But we have

$$y^2 < y^2 + y + 1 < (y+1)^2 = y^2 + 2y + 1, \qquad \text{if } y > 0$$

and

$$(y+1)^2 = y^2 + 2y + 1 < y^2 + y + 1 < y^2, \qquad \text{if } y < -1.$$

This means that unless $y(y+1) = 0$, the number $y(y+1) + 1$ is always between to consecutive squares so it cannot be equal to a square. This shows that the original product could only be zero.