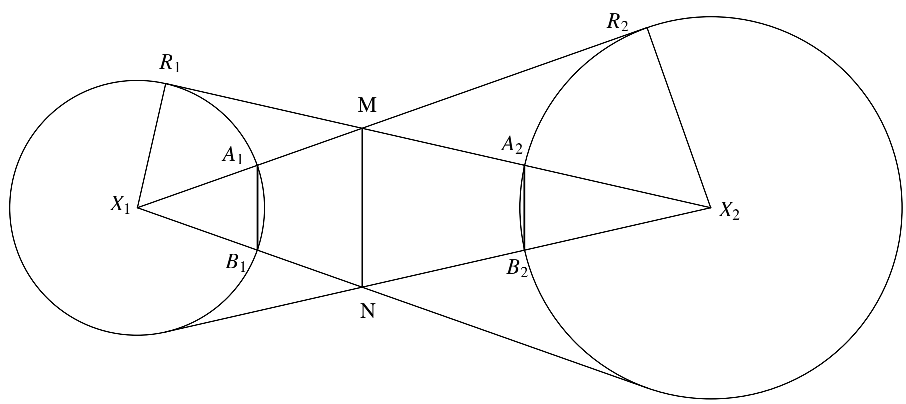
WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET V (2012-2013)

1. There are two non-intersecting circles C_1 and C_2, with centers X_1 and X_2 respectively, with neither circle inside the other. From X_1 and X_2, we draw the tangents to the opposite circles. For $i = 1, 2$, the tangents from X_i intersect C_i at points A_i and B_i. (We label the points so that A_1 and A_2 are on the same side of the line X_1X_2.) Show that the line segments A_1B_1 and A_2B_2 have equal length.

SOLUTION. Let lines X_1A_1 and X_2A_2 intersect at M, and lines X_1B_1 and X_2B_2 intersect at N. Let line X_2A_2 meet C_1 at R_1 and line X_1A_1 meet C_2 at R_2. Then triangles X_1R_1M and X_2R_2M are similar as they share a vertical angle at M and are both right triangles by the tangencies. So $X_1R_1/X_1M = X_2R_2/X_2M$. Since, $X_1R_1 = X_1A_1$ and $X_2R_2 = X_2A_2$, we have $X_1A_1/X_1M = X_2A_2/X_2M$. Triangles $X_1A_1B_1$ and X_1MN are similar by shared angle at X_1 and since they are both isosceles by symmetry across the line X_1X_2. So $X_1A_1/X_1M = A_1B_1/MN$. Similarly, $X_2A_2/X_2M = A_2B_2/MN$. This leads to $A_1B_1 = MN\frac{X_1A_1}{X_1M} = MN\frac{X_2A_2}{X_2M} = A_2B_2$.



2. Suppose that $a \geq b \geq c \geq 0$ and $a + b + c \leq 1$. Show that $a^2 + 3b^2 + 5c^2 \leq 1$.

SOLUTION. Since $a+b+c \geq 0$ we may square both sides of the inequality $a+b+c \leq 1$ to get

$$1 \geq (a + b + c)^2 \geq a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$$

Since $0 \leq c \leq b \leq a$ we have $ab \geq b^2$, $ac \geq c^2$, $bc \geq c^2$ which leads to

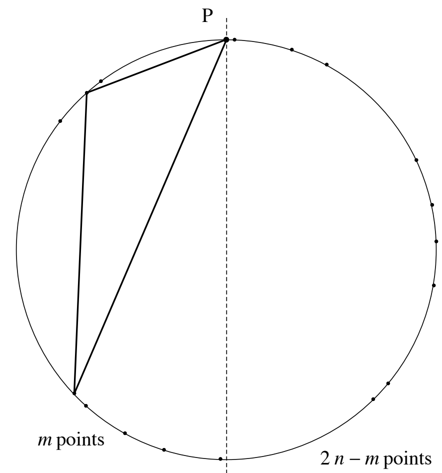
$$1 \geq a^2 + b^2 + c^2 + 2ab + 2ac + 2bc \geq a^2 + 3b^2 + 5c^2.$$

3. We would like to find sets $A_1, A_2, \ldots, A_n$ of size three which are all subsets of $\{1, 2, \ldots, 100\}$ and for any $1 \leq a < b \leq 100$ there is exactly one A_i with $\{a, b\} \subset A_i$. Decide if it is possible to construct such sets.

SOLUTION. We will show that this is not possible. Assume that there are sets $A_1, \ldots, A_n$ satisfying the conditions. Call an ordered pair (x, i) *good* if $2 \leq x \leq 100$ and $\{1, x\}$ is a subset of A_i. We will count the good pairs. Since for each x, we have that $\{1, x\}$ belongs to exactly one A_i, there are 99 good pairs. However, given an i that is the second coordinate in a good pair (x, i), we have that $A_i = \{1, x, y\}$ for some y. Thus (y, i) is also a good pair, and there are exactly two good pairs of which i is the second coordinate. Thus the number of good pairs must be even, a contradiction. The contradiction shows that it is not possible to find sets satisfying the conditions.

4. Given a set of $2n+1$ points on a circle, prove that there are at most $\frac{1}{6}n(n+1)(2n+1)$ acute triangles with vertices at those points.

SOLUTION. There are $\frac{1}{6}(2n+1)(2n)(2n-1)$ triangles since that's how many ways we can choose three points out of $2n+1$. Let P be one of the points on the circle, and suppose there are m of the points on one side of the diameter through P (including the point directly across from P in the m if it is in the set).



A triangle with vertices on the circle including at P is obtuse at a vertex other than P if its other two vertices are on the same side of the diameter through P because an obtuse angle subtends an arc greater than 180 degrees. A triangle with vertices on the circle including P and the point on the circle opposite P is right. So there are at least $\binom{m}{2}+\binom{2n-m}{2}$ obtuse or right triangles with an acute angle at P.

Consider $2n$ balls to be distributed into two boxes such that we wish to minimize the number of pairs of balls in the same box. If one box has m balls, and $m \geq n+1$, then by moving a ball from that box to the other, we lose $m-1$ pairs and gain $2n-m$ pairs, for a net loss of $2m-2n-1 \geq 1$ pairs. We can continue this process, each time lowering the number of pairs in the same box, until each box has n balls. Thus $\binom{m}{2}+\binom{2n-m}{2} \geq 2\binom{n}{2}$.

Each obtuse or right triangle has two acute angles. Doing the above at each point in the set, we count each obtuse or right triangle at most twice, and so the number of acute triangles is at most $\frac{1}{6}(2n+1)(2n)(2n-1)-(2n+1)\binom{n}{2} = n(n+1)(2n+1)/6$.

5. In a school we have n girl and n boy students with $n > 2013$. We know that the number of ways we can choose a club consisting of 5 boys and 6 girls is a square number. What's the smallest possible value of n?

SOLUTION. We can choose the 5 boys out of n exactly $\binom{n}{5} = \frac{n(n-1)(n-2)(n-3)(n-4)}{1\cdot2\cdot3\cdot4\cdot5}$ different ways and the 6 girls $\binom{n}{6} = \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{1\cdot2\cdot3\cdot4\cdot5\cdot6}$ different ways. Thus the number of possible choices for the club is

$$\binom{n}{5}\cdot\binom{n}{6} = \frac{n^2(n-1)^2(n-2)^2(n-3)^2(n-4)^2(n-5)}{1^2\cdot2^2\cdot3^2\cdot4^2\cdot5^2\cdot6} = \binom{n}{5}^2\cdot\frac{n-5}{6}$$

We know that this is a square so $\binom{n}{5}^2\cdot\frac{n-5}{6} = k^2$ with an integer k. Then $(6k)^2 = \binom{n}{5}^2 6(n-5)$ and since $\binom{n}{5}$ is an integer, this means that $6(n-5)$ must be a perfect square. Since it is divisible by 6, it has to be divisible by 36 as well and that means that $\frac{n-5}{6}$ is an integer and a perfect square. Since $n > 2013$ we get that $\sqrt{\frac{n-5}{6}} > \sqrt{\frac{2013-5}{6}} > 18$. Thus if $\frac{n-5}{6}$ is an integer and a perfect square then it is at least 19^2 which gives $6\cdot19^2+5 = 2171$ for the smallest possible value of n. (One can easily check that this will indeed work.)