WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET IV (2012-2013)

1. For any two points u and v in the plane, let $d(u,v)$ be the distance from u to v. Let $V = \{v_1, v_2, v_3, v_4, v_5\}$ be a set of five points inside a 2×2 square. Let S be the locus of points p in the plane so that the sum of the squares of the distances from p to each of the five point of V is 50, that is, S is the collection of those points p for which

$$d(p,v_1)^2 + d(p,v_2)^2 + d(p,v_3)^2 + d(p,v_4)^2 + d(p,v_5)^2 = 50.$$

Show that no matter how the five points in the square are chosen, the set S is a circle.

SOLUTION. Let us place the the 2×2 square in the Cartesian plane so that the origin is the lower left corner of square and the sides are on the respective axis. Denote the coordinates of the five points in V by $v_j = (x_j, y_j)$ for each $1 \le j \le 5$. Note that all of these coordinates are between 0 and 2. Let $\overline{x} = \frac{\sum_{j=1}^5 x_j}{5}$ and $\overline{y} = \frac{\sum_{j=1}^5 y_j}{5}$ and consider the point $\overline{v} = (\overline{x}, \overline{y})$. It is clear that $\bar{x}, \bar{y}$ are also between 0 and 2, so the point $\bar{v}$ is in the square. Note that for any number x we have

$$\sum_{j=1}^{5}(x - x_j)^2 = \sum_{j=1}^{5}\left(x^2 - 2xx_j + x_j^2\right) = \sum_{j=1}^{5}\left(x^2 - 2x\overline{x} + \overline{x}^2 + x_j^2 - 2x_j\overline{x} + \overline{x}^2\right)$$
$$= \sum_{j=1}^{5}\left((x-\overline{x})^2 + (x_j - \overline{x})^2\right) = 5(x-\overline{x})^2 + \sum_{j=1}^{5}(x_j - \overline{x}^2).$$

Similarly, $\sum_{j=1}^5 (y - y_j)^2 = 5(y-\overline{y})^2 + \sum_{j=1}^5 (y_j - \overline{y}^2)$. Then for any point $p = (x,y)$ in the plane,

$$\sum_{j=1}^{5} d(p,v_j)^2 = \sum_{j=1}^{5}\left((x-x_j)^2 + (y-y_j)^2\right) = 5d(p,\overline{v})^2 + \sum_{j=1}^{5} d(v_j,\overline{v})^2.$$

For each j, $d(v_j, \overline{v}) \le 2\sqrt{2}$, since both v_j and $\overline{v}$ are in the 2×2 square. Thus $\sum_{j=1}^5 d(v_j,\overline{v})^2 \le 5 \cdot 8 = 40$. Then S is the locus of points satisfying $d(p,\overline{v})^2 = \frac{50 - \sum_{j=1}^5 d(v_j,\overline{v})^2}{5} > 0$ which is the equation of a circle centered at $\overline{v}$. (The point $\bar{v}$ is actually the center of mass of the set V.)

2. At a party we have six married couples. The 12 guests sit down randomly around a large round table. What is the probability that at least one person is not sitting next to his/her spouse? (We assume that when they sit down to the 12 places, each configuration is equally likely.)

SOLUTION. We will count the number of all possible configurations (which we denote by A), and the number of configurations where there is at least one person not sitting next to his/her spouse (which we denote by B). Then the probability in question is B/A.

Let us number the guests from 1 to 12 so that the couples are $(1,2), (3,4), (5,6), (7,8), (9,10)$ and $(11,12)$. We can identify a possible configuration ('seating plan') by listing the number of the guests while we go around the table clockwise starting at 1. (To be precise, this way we identify those configurations that can be rotated into each other, but that will not change the probability in question.) When we list the guests in the seating chart, the first after 1 can be any of the other 11 guests, the next one can be any of the remaining 10 and so on. That means that we have $A = 11 \cdot 10 \cdot 9 \cdots 2 \cdot 1 = 11! = 39916800$.

In order to count the 'good' configurations we will actually count the 'bad' ones: those where each person sits next to his/her spouse. This will give us $A-B$ and using the value for A we will be able to compute the probability B/A. In a 'bad' configuration the two members of any couple sit next to each other, so we can identify such a configuration by first identifying the relative order of the six couples around the table (this can be done in $5! = 120$ different ways) and then identifying the position of the wife relative to the husband in each couple (which can be done 2 ways for each of the six couples). This means that the number of bad configurations is $A - B = 5! \cdot 2^6 = 7680$. Thus the probability in question is $\frac{11!-5!2^6}{11!} = \frac{10393}{10395}$.

3. The set A contains positive integer numbers. We know that if $x, y \in A$ and $x > y$ then $x - y \geq \frac{xy}{16}$. What is the maximal size of A?

SOLUTION. If $x > y > 0$ then $x - y \geq \frac{xy}{16}$ is equivalent to $\frac{1}{y} - \frac{1}{x} \geq \frac{1}{16}$. Thus if we take the reciprocals of the elements of A then the difference of any two of these numbers will be at least $\frac{1}{16}$. Then there cannot be two different integers in A which are at least 16, otherwise the difference of their reciprocals would be less than $1/16$. Also, since $\frac{1}{8} - \frac{1}{15} < \frac{1}{16}$, we cannot have more than one number in A from the set $\{8, 9, \ldots, 15\}$. Similarly, since $\frac{1}{5} - \frac{1}{7} < \frac{1}{16}$ we can only have at most one number in A from $\{5, 6, 7\}$. This means that A has at most three elements bigger than 4, so it cannot have more than 7 elements. But 7 can be achieved, it can be checked that e.g. $A = \{1, 2, 3, 4, 6, 10, 27\}$ will satisfy all the conditions.

4. Find all integers a, b, c such that $a^2 + b^2 + c^2 + 2abc = 0$.

SOLUTION. The only solution is $a = b = c = 0$. Suppose, for the sake of contradiction, that there is some other solution and let n be maximal such that 2^n divides a, b, and c. Then we have $a = 2^n a_0$ and $b = 2^n b_0$ and $c = 2^n c_0$, where a_0, b_0, c_0 are integers and at least one is odd. So $a_0^2 + b_0^2 + c_0^2 + 2^{n+1} a_0 b_0 c_0 = 0$. Since $a_0^2 + b_0^2 + c_0^2$ is even, we must have that exactly two of a_0, b_0, c_0 are odd, and so $a_0 b_0 c_0$ is even. Then $2^{n+1} a_0 b_0 c_0$ must be divisible by 4 and therefore the same is true for $a_0^2 + b_0^2 + c_0^2$. But this is a contradiction: the square of an odd number gives a remainder of 1 modulo 4 and the square of an even number is divisible by 4, so $a_0^2 + b_0^2 + c_0^2$ will give a remainder of 2 modulo 4.

5. Let ABC be an acute triangle with altitudes AD and CE. Show that the perpendicular from B to DE goes through the circumcenter of ABC.

SOLUTION.
Let F be the foot of the perpendicular from B to DE and denote the angle $\angle BAC$ by α. From right triangle ACE, we have $\angle ACE = 90° - \alpha$. Since $\angle AEC = 90° = \angle ADC$, we have that $AEDC$ lie on a circle. Thus $\angle EDC = 180° - \angle CAE = 180° - \alpha$ and $\angle FDB = \alpha$. By right triangle BDF, we have that $\angle CBF = \angle DBF = 90° - \alpha$. Let O be the circumcenter of ABC. We have that $\angle BOC = 2\angle BAC = 2\alpha$, and $OB = OC$ implies $\angle OBC = \angle OCB = 90° - \alpha$. Thus, $\angle CBF = \angle CBO$ which means that BFO are collinear.