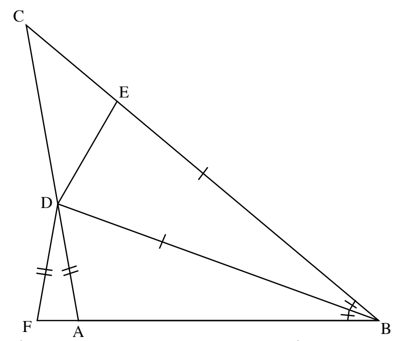
WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET III (2012-2013)

1. In the triangle ABC we have $AB = AC$ and $\angle BAC = 100°$. Let D be the intersection of the bisector at B and the side AC. Show that $BC = BD + DA$.

SOLUTION.



Let E be the point on the line segment BC for which $BE = BD$, and let F be the point on the line going through A and B where $DF = DA$ (with $A \neq F$). From this point the solution requires a bit of a book-keeping, but otherwise it is fairly straightforward. Since $AB = AC$, we have $\angle ABC = \angle BCA = 40°$. Since BD is a bisector, we have $\angle ABD = \angle DBE = 20°$. Since $BE = BD$, the triangle DBE is isosceles, and $\angle DEB = 80°$. But then $\angle DEC = 100°$, and the angles in DCE are $100°, 40°, 40°$. Thus $DE = EC$. The triangle ADF is isosceles (since $AD = DF$), and $\angle FAD = 80°$ (from $\angle CAB = 100°$). Then FAD has angles $80°, 20°, 80°$, and this means that the triangles FBD and DBE are congruent (their angles are the same and they have a common side), which means that $DF = DE$. Thus $DA = DF = DE = EC$, and this means that $BC = BE + EC = BE + AD = BD + DA$.

2. We have 100 points on a circle and connect every point with the other 99. What is the maximum number of intersection points we will have inside the circle?

SOLUTION. Let's call an intersection point inside the circle *simple* if there are exactly two line segments going through it. Note that any four points on the boundary will produce one intersection point: if we connect each with the other three, then we get exactly one intersection point inside the circle (the intersection of the diagonals of the formed quadrilateral). Since this way we will produce all the intersection points (some maybe multiple times), their number cannot be bigger than the number of ways we can choose four out of the 100 points, which is $\binom{100}{4} = \frac{100 \cdot 99 \cdot 98 \cdot 97}{4 \cdot 3 \cdot 2 \cdot 1} =$ 3,921,225. Moreover, we can reach this number if we can find 100 points on the circle so that all the intersections are simple. (Because in this case we counted each and every one exactly once.)

We can show that this is possible by induction on the number of points on the circle. The statement is certainly true for four points. In that case we have exactly one intersection point which is simple. Now assume that we can find n points on the circle so that all the intersection points are simple. Then there are finitely many places on the circle where the $(n+1)^{th}$ point would produce a non-simple intersection point (since one of the new line segments would need to pass through an existing simple intersection point). But there are infinitely many points on the circle, so we can easily find one which is different from the existing n points and does not produce a non-simple intersection point. This completes the proof that the maximum number is 3,921,225.

3. Is it possible to find an increasing sequence of 2012 positive integers so that the sum of any two consecutive numbers is equal to the square of the difference of those two numbers?

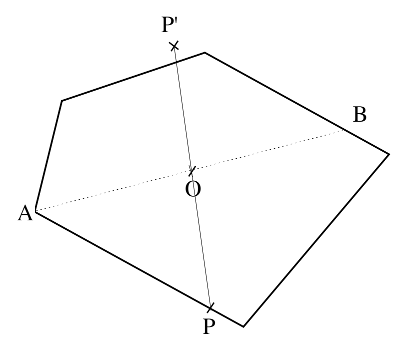
SOLUTION. The answer is yes, in fact one can find such a sequence with infinitely many numbers. An example of this is the sequence of triangular numbers: $1, 3, 6, 10, 15, 21, 28, \ldots$ where the n^{th}

term is $\frac{n(n+1)}{2}$ (the sum of the first n positive integers). Then the sum of two consecutive terms is $\frac{n(n+1)}{2} + \frac{(n+1)(n+2)}{2} = (n+1)^2$ and the difference of the two terms is $\frac{(n+1)(n+2)}{2} - \frac{n(n+1)}{2} = n+1$.

As we have seen, once we have the formula, it's easy to check that the sequence will work. But how do we find the formula? One possibility is to look at the equation $(b-a)^2 = a+b$ which the consecutive terms must satisfy. Solving this for b (and assuming $b > a$) we get $b = \frac{2a+1+\sqrt{8a+1}}{2}$. With the definition $f(x) = \frac{2x+1+\sqrt{8x+1}}{2}$ this means that if the first element of the sequence is c, then the subsequent elements are $f(c), f(f(c)), f(f(f(c))), \ldots$ Starting with $c = 1$ we can recognize the triangular numbers as the first couple of elements of the sequence, and once we have a guess for the formula it's easy to check that it works.
Note: one can actually show that any such sequence is just a subsequence of consecutive terms of the triangular number sequence.

4. Suppose that a convex polygon has a unit perimeter. Show that it can be covered with a disk of radius 1/4. (We do not assume that the polygon is regular!)

SOLUTION. Let A be one of the vertices of the polygon. If we 'walk' around the perimeter of the polygon, then the total length of the walk would be exactly one. Let B be the halfway point of the walk, and denote the midpoint of AB by O. We will show that a disk of radius 1/4, and center O covers the whole polygon.



Let's assume that the opposite is true: there is a point P on the boundary of the polygon which is not covered by the disk. This means that $OP > 1/4$. Consider the walk of length 1/2 on the perimeter from A to B which passes through P. Then the length of the walk between A to P is at least as big as the line segment AP (because the line segment gives the shortest way from A to B). Similarly: the length of the walk from P to B is at least as big as the line segment PB. Since the total length of the walk from A to B is 1/2, we get $AP+PB \leq 1/2$. Now consider the the reflection of P about the point O, and denote it by P'. Then $AP' = BP$ (because of the reflection), and in the triangle PAP' we have $PP' = 2OP > 1/2$ and $AP'+AP = BP+AP \leq 1/2$. But this is a contradiction because we cannot have $BP+AP < PP'$. This means that the initial assumption was incorrect, and the polygon is indeed covered by the disk.

5. In a ping-pong tournament we had 20 participants. Every player played with everybody else exactly once, and there were no ties. A player X gets a trophy if for every other player Y at least one of the following is true:
– X beat Y
– X beat somebody who beat Y.

Show that there will be at least one trophy awarded.

SOLUTION. Consider the player who had the most wins (if there are several players with the same number of wins, then just choose one of them). Let's call him Z. We will show that Z will receive a trophy.

Suppose that this is not true. Then there is a player Y who beat Z and also beat all the players that Z beat. But in that case Y would have beaten more players than Z which contradicts our choice of Z. This means that Z will indeed receive a trophy.