WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

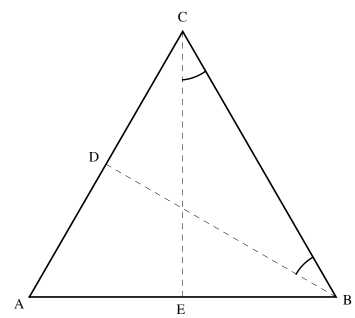
SOLUTIONS TO PROBLEM SET I (2012-2013)

1. Find all positive integers n such that n^3 and n^4 contain, between them, each of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 exactly once.

SOLUTION. The only such n is $n = 18$. Since $n^4 \geq n^3$, we see that n^4 contains at least 5 digits, and so $n \geq 10$. Since $n \geq 10$, we see that $n^4 = n \cdot n^3$ has at least one more digit than n^3, and thus n^3 has at most 4 digits. Since $22^3 = 10648$ has five digits, $n \leq 21$. Also, n^4 has at least 6 digits so $n \geq 18$ (as $17^4 = 83521$). We have that $18^3 = 5832$ and $18^4 = 104976$, so all the digits are used exactly once. However 19^4 contains the digit 1 twice (as the first and last digit), 20^3 has the digit 0 three times, and 21^3 and 21^4 both end with the digit 1.

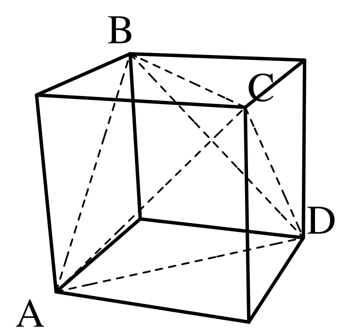
2. Let ABC be an acute triangle with median BD and altitude CE, where $BD = CE$ and $\angle DBC = \angle ECB$. Show that ABC is equilateral.

SOLUTION. We have that triangles DBC and ECB are congruent by side-angle-side. So $\angle ACB = \angle ABC$, and thus $AB = AC$. Also from the triangle congruence, $\angle BDC = \angle CEB = 90°$. Thus $\angle BDA = 90°$. Since BD is a median, $CD = DA$ and so triangles ADB and CDB are congruent by side-angle-side. Thus $AB = BC$. So all three sides of ABC are equal length.

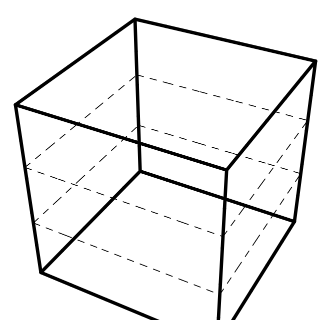


3. We color each point of a unit cube with one of three colors. Is it true that there are necessarily two points with the same color with distance at least 1.4? How about 1.5?

SOLUTION. We will show that one can always find two points with the same color with distance at least 1.4. Let A be one of the vertices of the cube, and let B, C and D be those vertices which are connected to A with a diagonal of one of the faces of the cube. Then any two of these four points will form the diagonal on one of the faces so their distance is $\sqrt{2} > 1.4$. (The four points form a regular tetrahedron.) Since we have four colors, two of the four points must have the same color and then their distance is at least 1.4.



Now we will show that we can color the points with three colors that the statement is not true with 1.5. To find an appropriate coloring first imagine that we divide the cube into three prisms with a unit square base and height 1/3. Coloring the three pieces with three different color will work: the distance of any two points in a given prism cannot be bigger than the length of the main diagonal which is $\sqrt{1^2 + 1^2 + (1/3)^2} = \sqrt{19/9} < 1.5 = \sqrt{9/4}$.



4. Let m, n be integers such that 23^{2011} divides $m^2 + n^2$. Show that 23^{2012} divides mn.

SOLUTION. We show by induction that if $23^{2k+1} | m^2 + n^2$ then $23^{2k+2} | mn$. (We write $a|b$ if a divides b.) The base case is $k = 0$. If we look at all the squares modulo 23, they give remainders $0, 1, 4, 9, 16, 2, 13, 3, 18, 12, 8, 6$, and pairing these up we can see that the only way for two squares to add to 0 modulo 23 is if they are both divisible by 23. This proves the $k = 0$ case, as $23|m^2$ implies $23|m$ and $23|n^2$ implies $23|n$ which yields $23^2|mn$. (We used that 23 is prime.) For general k, we apply the same reasoning to see that $23|m$ and $23|n$. Thus $23^{2k+1}|m^2 + n^2$ implies that $23^{2k-1}|(m/23)^2 + (n/23)^2$, which by induction implies $23^{2k}|(m/23)(n/23)$. But from this it follows that $23^{2k+2}|mn$, which is what we wanted to prove.

5. Consider 16 lattice points arranged on a 4×4 square grid. We color the first point of the third row black and the other 15 points with white. Next in each step we can choose a horizontal or a vertical line or a line which is parallel to one of the main diagonals and we can change the colors of the lattice points on that line to the opposite. Is it possible to change the colors of all the lattice points to white?

SOLUTION. We will prove that this is not possible. Denote by A the set of those points which are on the side of the 4×4 grid, but not one of its corners. We will show that we cannot change the color of all the points in A to white, as we will always have an odd number of white points there. This is certainly true at the beginning as we have one black and seven white points in A. If we show that after each step the parity of white points in A stays the same then this will finish the proof. Note, that any line that we consider will intersect the set A in zero or two points. In the first case we do not change the colors in A. In the second case the number of white points can only change if there were two black or two white points in the intersection and then the change is ± 2. This shows that the parity of the white points in A will never change, which is what we wanted to show.