WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET V (2011-2012)

1. How many dice should we roll if we want to make the probability of having exactly two sixes as large as possible?

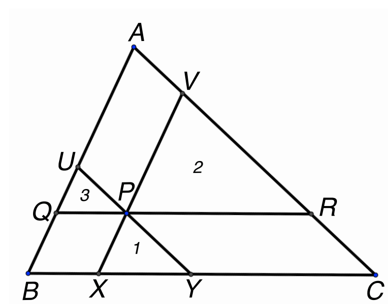
SOLUTION. Let p_n denote the probability that we get exactly two sixes if we roll n dice. Clearly, $p_1 = 0$ and $p_2 > 0$. If $n \geq 2$ then there are 6^n possible outcomes, since each die can produce any number between 1 and 6, independently of the others, and each such outcome will have the same probability. Next, we count the outcomes where we get exactly 2 sixes. To obtain such an outcome, we first specify the two dice that will show sixes and there are $\binom{n}{2} = \frac{n(n-1)}{2}$ such choices for these. Once these two dice are specified, there are then 5^{n-2} possible outcomes since each of the remaining $n-2$ die can now produce any number between 1 and 5, independently of each other. Thus there are $\frac{n(n-1)}{2}5^{n-2}$ different ways we can get exactly two sixes, so $p_n = \frac{n(n-1)}{2}5^{n-2}6^{-n}$.

Now we need to find n for which p_n is maximal. To do this, we first compute the ratio p_{n+1}/p_n for $n \geq 2$, which simplifies to

$$\frac{p_{n+1}}{p_n} = \frac{\frac{(n+1)n}{2}5^{n-1}6^{-n-1}}{\frac{n(n-1)}{2}5^{n-2}6^{-n}} = \frac{n+1}{n-1} \cdot \frac{5}{6} = \frac{5}{6}\left(1 + \frac{2}{n-1}\right).$$

This ratio is bigger than one if $2 \leq n \leq 10$, equal to one if $n = 11$ and less than one if $n \geq 12$. Thus $p_n < p_{n+1}$ if $2 \leq n \leq 10$, $p_{11} = p_{12}$ and $p_n > p_{n+1}$ if $n \geq 12$. It follows that the maximal probability is achieved at 11 and 12. The numerical value of $p_{11} = p_{12}$ is approximately 0.296094.

2. In the figure, point P lies in the interior of $\triangle ABC$, and lines parallel to the sides of the triangle have been drawn through P. These lines decompose the original triangle into three quadrilaterals and three triangles, and we label the three small triangles 1, 2 and 3, as shown. Let a be the area of $\triangle ABC$, and write a_1, a_2 and a_3 to denote the areas of triangles 1, 2 and 3, respectively. Prove that $\sqrt{a} = \sqrt{a_1} + \sqrt{a_2} + \sqrt{a_3}$.



SOLUTION. Label points Q, R, X, Y, U and V as shown, so that $\triangle XPY$, $\triangle PVR$ and $\triangle QUP$ are respectively triangles 1, 2 and 3. Because $\overline{XV}$ is parallel to $\overline{BA}$ and $\overline{YU}$ is parallel to $\overline{CA}$, we see that $\angle PXY = \angle B$ and $\angle PYX = \angle C$, and thus $\triangle XPY$ is similar to $\triangle BAC$ by the Angle-Angle similarity criterion. In the same way, we see that $\triangle QUP$ is similar to $\triangle BAC$ and $\triangle PVR$ is similar to $\triangle BAC$.

In general, the ratio of the areas of two similar triangles is equal to the square of the ratio of each pair of corresponding sides. In this problem, therefore, we have $a_1/a = (XY/BC)^2$, $a_2/a = (PR/BC)^2$ and $a_3/a = (QP/BC)^2$. It follows that

$$\sqrt{\frac{a_1}{a}} + \sqrt{\frac{a_2}{a}} + \sqrt{\frac{a_3}{a}} = \frac{XY + PR + QP}{BC} = \frac{XY + YC + BX}{BC} = \frac{BC}{BC} = 1,$$

where the second equality holds because $PR = YC$ and $QP = BX$ since $YPRC$ and $BQPX$ are parallelograms. Multiplication by $\sqrt{a}$ yields $\sqrt{a_1} + \sqrt{a_2} + \sqrt{a_3} = \sqrt{a}$, as wanted.

3. We have 100 piles of coins, with the i^{th} pile containing exactly i coins. We wish to remove all the coins in a series of steps. In each step we are allowed to take away coins from as many piles as we wish, but we have to take the same number of coins from each such pile. Show that we can remove all the coins in 7 steps, but that it is impossible to do so in 6.

SOLUTION. We first show that seven steps is sufficient by using the following procedure. In each step, if the largest pile has $2k$ or $2k-1$ coins, then we remove k coins from each pile of size at least k. This will result in piles of size at most k or $k-1$, respectively. In the first step we remove 50 coins from piles 50 to 100, and thus we are left with piles of size at most 50. In the next few steps, we change the maximum pile size from 50 to 25, then to 12, then 6, 3, and 1. After the sixth step we only have piles with single coins, and we can remove those in one more step.

Now suppose that we can remove all the coins in n steps. We will show that $n \geq 7$. For this, let us denote by a_k the number of coins removed from each chosen pile in the k^{th} step. Since the i^{th} pile is removed in the end for any $1 \leq i \leq 100$, each such number i can be expressed as a sum of certain of the numbers $a_1, a_2, \ldots, a_n$. Now the number of such sums is at most 2^n since, for each $k = 1, 2, \ldots, n$, we can either include a_k in the sum or not. Thus 2^n must be at least as big as 100, and therefore n is at least 7.

4. Find all integers n such that $n^2 - 13n + 50$ is a perfect square, and prove that you have found them all.

SOLUTION. If $n^2 - 13n + 50$ is the square of an integer, we can write $n^2 - 13n + 50 = (n-r)^2$ for some integer r, which may be negative. Subtracting n^2 from both sides, we get $-13n + 50 = -2nr + r^2$, and thus $n(2r-13) = r^2 - 50$, and we have $n = (r^2-50)/(2r-13)$. Now observe that $(2r-13)(2r+13) = 4r^2 - 169 = 4(r^2-50) + 31$. Division by $2r-13$ then yields

$$2r + 13 = \frac{4(r^2-50)}{2r-13} + \frac{31}{2r-13} = 4n + \frac{31}{2r-13},$$

and thus $31/(2r-13)$ is an integer. Since 31 is prime, it follows that $2r-13$ must be one of 1, -1, 31 or -31, and so r is one of 7, 6, 22 or -9. The corresponding values of $n = (r^2-50)/(2r-13)$ are respectively -1, 14, 14 and -1, and so the only possibilities for n are -1 and 14. If $n = -1$, then $n^2 - 13n + 50 = 64$, and if $n = 14$, we also get $n^2 - 13n + 50 = 64$. Since 64 is a square, these two values of n solve the problem.

5. Show that if a and b are real numbers bigger than $\dfrac{1}{2}$ then

$$a + 2b - 5ab < \frac{1}{4}.$$

SOLUTION. Since a and b are real numbers bigger than $1/2$, we can write $a = x + 1/2$ and $b = y + 1/2$ with both x and y positive. Then

$$\begin{aligned} a + 2b - 5ab - \frac{1}{4} &= (x+1/2) + 2(y+1/2) - 5(x+1/2)(y+1/2) - \frac{1}{4} \\ &= -\frac{3x}{2} - \frac{y}{2} - 5xy < 0, \end{aligned}$$

so clearly $a + 2b - 5ab < 1/4$.