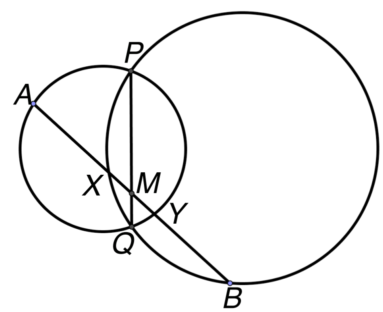
WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET IV (2011-2012)

1. Suppose that a rectangular array of numbers has the following two properties. First, there is some constant A such that in every row, the sum of the largest and smallest numbers is A, and second, there is some constant B such that in every column, the sum of the largest and smallest numbers is B. Prove that $A = B$.

SOLUTION. Let l and s, respectively, be the largest and smallest numbers in the array. Then l is certainly the largest number in its row, and thus $A - l$ is the smallest number in that row. It follows that $s \le A - l$ because s is the smallest number in the whole array, and thus $l + s \le A$. Also, s is the smallest number in its row, and hence $A - s$ is the largest number in that row, and we have $l \ge A - s$ because l is the largest number in the whole array. Then $l + s \ge A$, and since we saw that also $l + s \le A$, we deduce that $l + s = A$. Similar reasoning, working with columns in place of rows, yields $l + s = B$, and we conclude that $A = B$.

2. In the figure, line $\overline{AB}$ joins point A on one circle to point B on another circle, and X and Y are the other two points where $\overline{AB}$ meets the circles, as shown. Assume that the common chord $\overline{PQ}$ passes through the midpoint M of $\overline{AB}$. Prove that M is also the midpoint of $\overline{XY}$.



SOLUTION. Recall that when two chords of a circle cross, dividing each chord into two pieces, the product of the lengths of the two parts of one chord is equal to the product of the lengths of the two pieces of the other chord. Now $\overline{AY}$ and $\overline{PQ}$ are two crossing chords of the left circle in the figure, and thus $MA{\cdot}MY = MP{\cdot}MQ$. (Since $\triangle MAQ$ and $\triangle MPY$ are similar). Similarly, looking at the right circle in the picture, we have $MB{\cdot}MX = MP{\cdot}MQ$. Then $MA{\cdot}MY = MP{\cdot}MQ = MB{\cdot}MX$, and since $MA = MB$, we can cancel to get $MY = MX$, as required.

3. In a group of 30 students, each student knows exactly six of the remaining 29 students. Of course, A knows B if and only if B knows A. We will call a group of <u>three</u> students *balanced* if either all three students know each other or no one knows anyone else within the group. In how many ways can we choose a balanced group?

SOLUTION. There are $\binom{30}{3} = (30{\cdot}29{\cdot}28)/(3{\cdot}2{\cdot}1) = 4060$ ways to choose three students out of the 30, so this is the total number of groups of three. We will count the number of these groups that are not balanced. Note that in a non-balanced group there are exactly two students whose relationships with the other two members of the group are different (i.e. he/she knows one, but not the other) and in a balanced group we cannot have such a student. This means that if we count the number of ordered triples (A, B, C) of students where A knows B, but not C, then this will be exactly twice the number of non-balanced groups. But counting these triples is easy. To start with, we can choose A in 30 ways. Furthermore, for each choice of A, there are 6 choices for B and 23 choices for C (independently of each other). This means that the total number of these triples is $30{\cdot}6{\cdot}23 = 4140$ and the number of balanced groups is $4060 - (4140/2) = 1990$.

Although it is not part of the problem, it is easy to see that we can have a situation where each of the 30 students knows exactly six others.

4. Find the maximum of the expression

$$x^4y + x^3y + x^2y + xy + xy^2 + xy^3 + xy^4$$

if x, y are real numbers with $x + y = 2$.

SOLUTION. Since $x + y = 2$, we see that

$$x^2 + y^2 = (x + y)^2 - 2xy = 4 - 2xy$$

and

$$x^3 + y^3 = (x + y)[(x^2 + y^2) - xy] = 2[(4 - 2xy) - xy] = 8 - 6xy\,.$$

Thus, by rearranging terms, we have

$$\begin{aligned}
&x^4y + x^3y + x^2y + xy + xy^2 + xy^3 + xy^4 \\
&\qquad = xy[(x^3 + y^3) + (x^2 + y^2) + (x + y) + 1] \\
&\qquad = xy[(8 - 6xy) + (4 - 2xy) + 2 + 1] = xy[15 - 8xy]\,.
\end{aligned}$$

Note that the latter function of the variable xy is an upside-down parabola and we know that the maximum value occurs at the average of the two roots. Alternatively, it follows from

$$xy[15 - 8xy] = \frac{225}{32} - 8\left(xy - \frac{15}{16}\right)^2$$

that the maximum value of $225/32$ will be achieved when $xy = 15/16$. But there remains the question of whether $xy = 15/16$ can occur for real numbers x and y with $x + y = 2$. Solving the equation $x(2 - x) = 15/16$, we get $x = 3/4$ and $x = 5/4$ which shows that the answer is "yes".

5. (New Year's Problem) Show that there exist integers X and Y such that the product

$$P = (1^2 + 2^2)(2^2 + 3^2)(3^2 + 4^2) \cdots (2011^2 + 2012^2)$$

is equal to $X^2 + Y^2$.

SOLUTION. For convenience, let us say that an integer N is *special* if $N = A^2 + B^2$ for two integers A and B. We note that

$$(a^2 + b^2)(c^2 + d^2) = (ad + bc)^2 + (ac - bd)^2.$$

Furthermore, if a, b, c and d are integers, then so are $(ad + bc)$ and $(ac - bd)$. It follows that a product of two special integers is special and hence, by induction, a product of a finite number of special integers is again special. Since the factors $[n^2 + (n + 1)^2]$ of P are all special, we conclude from the above that P is also special.