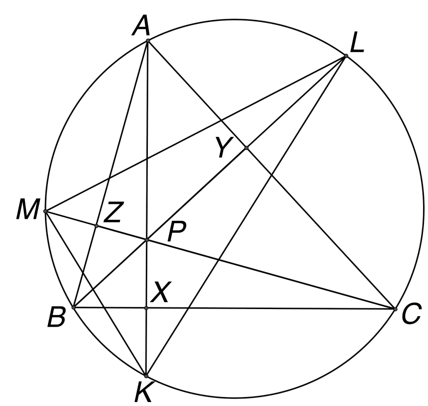
WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET III (2011-2012)

1. We place 12 coins on the squares of an 8×8 chessboard (we might put more than one coin on a given square). Show that we can erase four rows and four columns from the board so that the remaining 16 squares do not contain any coins.

SOLUTION. Count the number of coins in each row and suppose that the numbers we obtain are $a_1 \leq a_2 \leq \cdots \leq a_8$. Clearly, $a_1 + a_2 + \cdots + a_8$ is the total number of coins, namely 12. We will show that $a_1 + a_2 + a_3 + a_4 \leq 4$. To prove this inequality, we consider two cases. Suppose first that $a_5 \geq 2$. Since $2 \leq a_5 \leq a_6 \leq a_7 \leq a_8$, we have $8 \leq a_5 + a_6 + a_7 + a_8$ and hence there are at most $12 - 8 = 4$ coins remaining. On the other hand, if $a_5 \leq 1$, then $a_1 \leq a_2 \leq a_3 \leq a_4 \leq 1$ and thus again $a_1 + a_2 + a_3 + a_4 \leq 4$. Now erase the rows corresponding to a_5, a_6, a_7 and a_8. That is, we erase the four rows with the largest number of coins in them. As we have shown, there are at most four coins in the remaining four rows. By choosing four columns to cover these remaining coins, we can be sure that the chosen rows and columns cover all of our coins.

2. In the figure at the right, lines $\overline{AX}$, $\overline{BY}$ and $\overline{CZ}$ have been drawn in $\triangle ABC$, and point P lies on each of these three lines. Also, $\angle XAY = \angle XBY$ and $\angle XAZ = \angle XCZ$. Prove that $\angle ZBY = \angle ZCY$.



SOLUTION. Draw the circle through points A, B and C and extend lines $\overline{AX}$, $\overline{BY}$ and $\overline{CZ}$ to meet the circle at points K, L and M, respectively, and then draw $\triangle KLM$, as shown. Equal angles $\angle XAY$ and $\angle XBY$ are inscribed angles subtending arcs $\overparen{KC}$ and $\overparen{CL}$ respectively, and thus these arcs are equal in degrees. Since $\angle CMK$ and $\angle CML$ also subtend these equal arcs, it follows that $\angle CMK = \angle CML$, and thus $\overline{MC}$ is the bisector of $\angle M$ in $\triangle KLM$. Similarly, $\overline{LB}$ is the bisector of $\angle L$ in $\triangle KLM$, and both of these bisectors go through P.

Since the three angle bisectors of a triangle go through a common point, it follows that P must also lie on the bisector of $\angle K$ in $\triangle KLM$, and so $\overline{KA}$ bisects $\angle K$, and we have $\angle MKA = \angle LKA$. We conclude that arcs $\overparen{MA}$ and $\overparen{LA}$ are equal in degrees. Since $\angle ZBY$ and $\angle ZCY$ subtend these equal arcs, we conclude that $\angle ZBY = \angle ZCY$, as wanted.

3. How many solutions, in positive integers x and y, are there for the equation $x^2 = y^2 + 4 \cdot 3^{2011}$.

SOLUTION. Clearly $x > y > 0$ and $(x - y)(x + y) = x^2 - y^2 = 4 \cdot 3^{2011}$. Moreover, the latter product is equivalent to the original equation. Notice that $(x + y) = (x - y) + 2y$, so these factors of $4 \cdot 3^{2011}$ are either both even or both odd. But certainly they cannot be both odd since their product is even. It follows that they are both even and hence each is 2 times a power of 3, say $x - y = 2 \cdot 3^n$ and $x + y = 2 \cdot 3^{2011-n}$ for some nonnegative integer n. Since $x + y > x - y$, we have $2011 - n > n$, and thus $0 \leq n \leq 1005$. Now for each of these 1006 possibilities for n, we have $x - y = 2 \cdot 3^n$ and $x + y = 2 \cdot 3^{2011-n}$, so solving for x and y yields $x = 3^{2011-n} + 3^n$ and $y = 3^{2011-n} - 3^n$, and these are all positive integer solutions to the original equation. Thus there are precisely 1006 such solutions.

4. Let a and $x_1, x_2, \ldots, x_n$ be positive numbers, and suppose that the sum of the x_i exceeds a. For each subset J of $\{1, 2, \ldots, n\}$, write s_J to denote the sum of those numbers x_j for which j is in J, and note that $s_J = 0$ if J is the empty set. Assume also that whenever $s_J < a$, we have $x_i \le a - s_J$ for all subscripts i not in J. Show that all x_i are equal and that a is an integer multiple of their common value.

SOLUTION. Write $I = \{1, 2, \ldots, n\}$, and observe that s_I is the sum of all x_i, so $s_I > a$, by assumption. Choose a subset K of I, where K is as small as possible such that $s_K \ge a$. Note that K is not the empty set since $a > 0$. Let k be an arbitrary member of K, and consider the set J (which may be empty) obtained by deleting k from K. We conclude from the minimality of K that $s_J < a$. Since k is not in J, it follows by assumption that $x_k \le a - s_J$. Thus $s_K = s_J + x_k \le a$. Since we chose K so that $s_K \ge a$, we deduce that $s_K = a$. In particular, K is not the whole set I since $s_I > a$.

To prove that all x_i are equal, we show that $x_k = x_i$ whenever k is in K and i is not in K. As in the previous paragraph, let J be the set that results by deleting k from K. Since i is not in J, we have $x_i \le a - s_J = s_K - s_J = x_k$. To prove that $x_i = x_k$, we suppose that $x_i < x_k$, and we derive a contradiction. Let L be the set obtained by adjoining i to J. Then $s_L = s_J + x_i = s_K - x_k + x_i < s_K = a$. Since k is not in L, we have $x_k \le a - s_L = s_K - (s_K - x_k + x_i) = x_k - x_i$, and thus $x_i \le 0$, which is false. Therefore $x_k = x_i$, as wanted. Finally, let x be the common value of all x_i then $a = s_K = mx$, where m is the number of members of K.

5. If x, y and z are positive numbers, prove that

$$3\Big(\frac{x+y}{z} + \frac{y+z}{x} + \frac{z+x}{y}\Big) \ge 2(x+y+z)\Big(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\Big)\,.$$

SOLUTION. To simplify the notation, let us write $A = \big(\frac{x+y}{z} + \frac{y+z}{x} + \frac{z+x}{y}\big)$ and $B = (x+y+z)\big(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\big)$. We must show, therefore, that $3A \ge 2B$. Adding $z/z + y/y + x/x = 3$ to A, we obtain

$$A + 3 = \Big(\frac{x+y+z}{z} + \frac{x+y+z}{x} + \frac{x+y+z}{y}\Big) = (x+y+z)\Big(\frac{1}{z} + \frac{1}{y} + \frac{1}{x}\Big) = B.$$

Now observe that $0 \le (x-y)^2 = x^2 - 2xy + y^2$, so $2xy \le x^2 + y^2$, and division by the positive number xy yields $2 \le x/y + y/x$. Similarly, $2 \le x/z + z/x$ and $2 \le y/z + z/y$. It follows that

$$\begin{aligned} B &= (x+y+z)\Big(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\Big) \\ &= 1 + 1 + 1 + \Big(\frac{x}{y} + \frac{y}{x}\Big) + \Big(\frac{x}{z} + \frac{z}{x}\Big) + \Big(\frac{y}{z} + \frac{z}{y}\Big) \ge 3 + 2 + 2 + 2 = 9\,. \end{aligned}$$

With this, we see that $3A = 3(B-3) = 2B + B - 9 \ge 2B$, as wanted.