WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

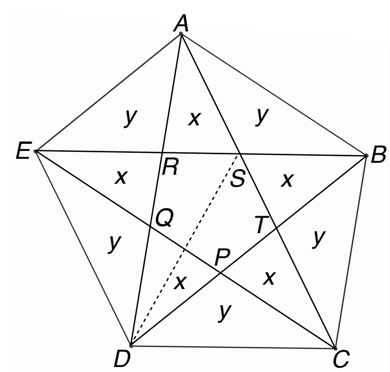
SOLUTIONS TO PROBLEM SET II (2011-2012)

1. We are given 13 (positive) integer weights and we know that if we discard any one of them then we can divide the rest into two groups of six so that the total weights are the same in the two groups. Show that all the weights must be equal.

SOLUTION. We will say that the integer weights are *balanced* if they have the property described in the problem. Assume by way of contradiction that there exists a counterexample where the weights are not all equal. Consider all such counterexamples and choose one where the total sum of the weights is as small as possible. Denote the weights by a_i for $1 \le i \le 13$. According to the balanced condition, if we leave out any of the a_i's, the rest can be divided into two groups of six with the same sum. In particular, the sum of the remaining 12 weights is even. This means that each a_i will have the same parity as $S = a_1 + a_2 + \cdots + a_{13}$. Therefore either all the a_i's are even or all the a_i's are odd. We will show that this leads to a contradiction by constructing another counterexample with the sum being strictly smaller than S.

If all the a_i's are even, then the integers $b_i = a_i/2$ will work. Indeed, the total sum is $S/2 < S$, the new integers b_i are still not all equal, and the b_i's will also be balanced. On the other hand, if all the a_i's are odd and they are not all equal, then their sum is bigger than 13. If we now consider the integers $b_i = (a_i + 1)/2$, then these are also balanced and they are not all equal. Furthermore, their total sum is $(S + 13)/2$. Since $S > 13$ we have $(S + 13)/2 < S$, and we have again found a counterexample with a smaller total sum than S. Thus our original assumption was incorrect, so we conclude that if the thirteen weights are balanced, then they all must be equal.

2. A pentagon is divided into a smaller pentagon and ten triangles, as in the diagram. Of the ten triangles, five have area x and five have area y, as indicated. Compute the ratio y/x.



SOLUTION. Label the points as shown, and draw line $\overline{DS}$. Since $\triangle BAS$ and $\triangle BCT$ have equal areas and equal heights with respect to bases $\overline{AS}$ and $\overline{CT}$, we see that $AS = CT$. Now look at $\triangle DAS$ and $\triangle DCT$, viewing $\overline{AS}$ and $\overline{TC}$ as their bases. These triangles have equal heights and equal bases, and so they have equal areas. The area of $\triangle DTC$ is $x + y$, so the same is true of $\triangle DAS$, and since the area of $\triangle ARS$ is equal to x, it follows that the area of $\triangle DRS$ is equal to y.

Now $\triangle AES$ is divided by $\overline{AR}$ into triangles with areas y and x, so it follows that point R divides the base $\overline{ES}$ into two pieces with ratio y/x, and we have $y/x = ER/RS$. Line $\overline{ES}$ can also be viewed as the base of $\triangle DES$, and this triangle is divided into two parts by line $\overline{DR}$. Since we have established that the area of $\triangle DRS$ is y, and we know that the area of $\triangle DRE$ is $x + y$, we see that $ER/RS = (x + y)/y$. We saw previously that this ratio is equal to y/x, and we deduce that $y/x = (x + y)/y$. By cross-multiplying, we get $y^2 = x^2 + xy$, and thus $y^2 - xy - x^2 = 0$.

We must determine the ratio $r = y/x$. We have $y = xr$, so substituting into the equation $y^2 - xy - x^2 = 0$, we get $r^2x^2 - rx^2 - x^2 = 0$, and thus $r^2 - r - 1 = 0$. This yields $r = (1 \pm \sqrt{5})/2$, and since r is clearly positive, we see that $r = (1 + \sqrt{5})/2$. Note, for example, that if the pentagon is regular, then the appropriate triangles do indeed have equal areas.

3. Suppose that $f(x)$ is a function that is defined for all rational numbers x, and assume that $f(x) = f(x-2)$ for all such x. Assume also that if x is nonzero, then $f(x) = f(1/x)$. Show that for every rational x, either $f(x) = f(0)$ or $f(x) = f(1)$.

SOLUTION. Write $x = a/b$, where a and b are integers, $b \geq 1$, and the fraction a/b is reduced to lowest terms. We prove the result by induction on b. If $b = 1$, then x is an integer, and since we can shift x by 2 in either direction without changing $f(x)$, we can reduce to the case where $x = 0$ or $x = 1$, and we are done.

Now let $b > 1$, so that x is not an integer, and assume that the result holds for all rational numbers with positive denominator smaller than b. Repeatedly shifting by 2 or -2, we can clearly assume that $-1 < x < 1$, and these shifts do not change the denominator. We can therefore write $x = a/b$, where this fraction is in lowest terms and $-b < a < b$. Since x is not an integer, it is nonzero, so $f(x) = f(1/x)$. Furthermore, the denominator of $1/x$ is $|a| < b$, so by the inductive hypothesis, $f(1/x)$ is either $f(0)$ or $f(1)$.

4. Suppose we have two urns, each containing some red and some blue marbles, with at least one of each color in each urn. Assume that if we choose an urn randomly and then choose a marble randomly from that urn, then the probability of picking a red marble is the same as we would get by combining all the marbles into one urn and choosing a marble from that one at random. If the first urn contains 7 marbles and the second one contains 5 red marbles, how many marbles can there be in the second urn?

SOLUTION. Denote the number of red marbles in the first urn by r and the total number number of marbles in the second by t. If we choose a marble randomly from the first urn then the probability of getting a red one is $r/7$, while this probability is $5/t$ for the second urn. Now we begin by choosing our urn at random, so we have $1/2$ probability of choosing from the first and $1/2$ of choosing from the second urn. This means that the probability of ending up with a red marble is $\frac{1}{2}(\frac{r}{7} + \frac{5}{t})$. On the other hand, if we combine all the marbles into one urn, then we have $r+5$ red marbles out of a total $7+t$, so the probability of getting a red marble in this case is $\frac{r+5}{7+t}$. If these two probabilities are equal then

$$0 = \frac{1}{2}\left(\frac{r}{7} + \frac{5}{t}\right) - \frac{r+5}{7+t} = \frac{245 - 35t - 7rt + rt^2}{14t(7+t)} = \frac{(t-7)(rt-35)}{14t(7+t)}.$$

The expression on the right can only be zero if $t = 7$ or $rt = 35$. Now $1 \leq r \leq 6$, since both colors are represented in the first urn, so the second equation yields $r = 1, t = 35$ or $r = 5, t = 7$. Therefore, the only possibilities for t are 7 and 35.

Thus we can have 7 marbles in the second urn (in which case the number of red marbles in the first urn can be any value between 1 and 6), or we can have 35 marbles in the second urn (in which case there is exactly 1 red in the first urn).

5. Find all prime numbers $p < q$ such that $p^2q^2 + 5$ is also prime.

SOLUTION. If both p and q are odd, then $p^2q^2 + 5$ is even and larger than 2. Thus $p^2q^2 + 5$ cannot be a prime. If follows that p, the smaller of p and q, must equal 2. Next, if $q \neq 3$, then both p and q leave a remainder of 1 or 2 when divided by 3, and hence both p^2 and q^2 leave a remainder of 1 when divided by 3. Thus $p^2q^2 + 5$ is divisible by 3, and so it is not prime. Thus we must have $q = 3$, and we observe that $2^2 3^2 + 5 = 41$ is indeed prime.