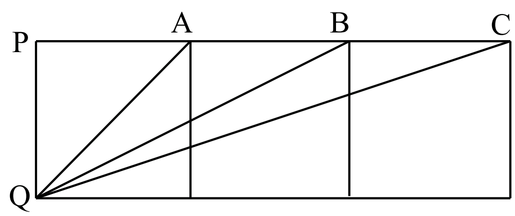
WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET I (2011-2012)

1. Show that if we have 100 distinct points on the plane, then we can find a line on the plane so that exactly 50 of the 100 points lie on each side of the line.

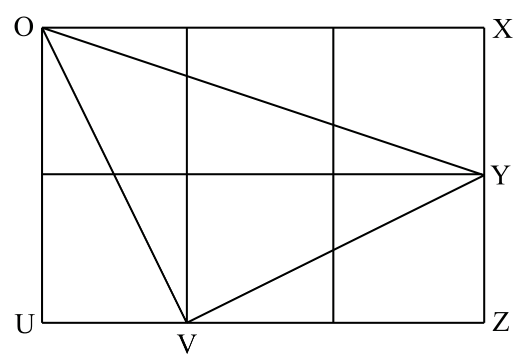
SOLUTION. Consider all the lines going through at least two of these points. We have finitely many such lines (their number is at most $(100{\cdot}99)/2$), so we can find a "special" line t that is not parallel to any of these. This means that any line parallel to t can only pass through at most one of the 100 points. For convenience, let us rotate the plane so that t is horizontal. Now, draw a line parallel to t through every one of the 100 points. In this way, we get 100 distinct horizontal lines on the plane and we number them from 1 to 100 in order, starting from the lowest line and ending at the highest one. Now draw a horizontal line that is between the 50^{th} and the 51^{st} line. This will have exactly 50 of the original lines above it and 50 below. Thus this new line will have exactly 50 of the 100 points above it and the remaining 50 below.

2. The picture below shows three adjacent unit squares. Determine the sum of the three angles $\angle PAQ, \angle PBQ$ and $\angle PCQ$.



SOLUTION. The sum is $90°$. We will prove this by dividing a right angle into three pieces, corresponding to each of the three angles. It is clear that $\angle PAQ = 45°$ since $\triangle PAQ$ is an isosceles right triangle. Now consider the second picture consisting of six unit squares.

We have $\angle UOV = \angle PBQ$ and $\angle XOY = \angle PCQ$ since the corresponding right triangles are congruent, that is $\triangle UOV \cong \triangle PBQ$ and $\triangle XOY \cong \triangle PCQ$. We now prove that $\angle VOY$ is $45°$, which will show that $\angle VOY = \angle PAQ$. Note that the right triangles $\triangle UOV$ and $\triangle ZVY$ are congruent, so $\angle ZVY + \angle UVO = \angle ZVY + \angle ZYV = 90°$. This yields $\angle OVY = 180° - 90° = 90°$. Since $OV = VY$, this means that $\triangle OVY$ is an isosceles right triangle, and therefore $\angle VOY = 45°$. Thus $\angle PAQ + \angle PBQ + \angle PCQ = \angle UOV + \angle VOY + \angle YOX = \angle UOX = 90°$.



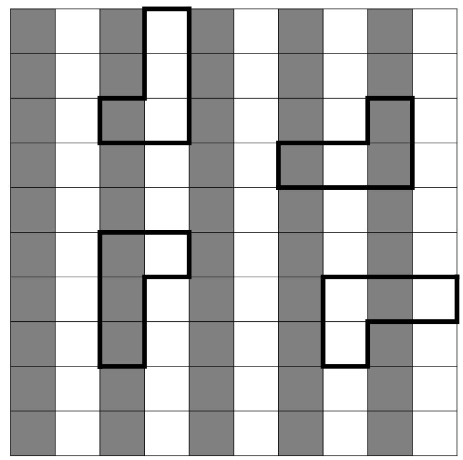
3. Suppose that x, y and z are integers with the property that no integer larger than 1 exactly divides all of them. Let

$$A = x + 2y\,, \quad B = y + 2z\,, \quad \text{and} \quad C = z + 2x\,.$$

Find the largest integer that can exactly divide all of A, B and C.

SOLUTION. Suppose that m divides A, B and C. Assume first that m is divisible by some prime number p that also divides z. Since p divides $B = y + 2z$, it follows that p divides y. Also, p divides $A = x + 2y$, and thus p divides x. Now $p > 1$ and p divides each of x, y and z, a contradiction. We conclude, therefore, that there cannot be any prime divisor of m that divides z. Now m divides $C - 2A = z - 4y$, and thus m divides $4B + (z - 4y) = 9z$. But no prime divisor of m divides z, and thus m must divide 9. It follows that the largest number that can divide all of A, B and C is at most 9. Finally, we show that 9 really can divide all three of A, B and C. To see this, take $x = 2$, $y = 8$ and $z = 5$. Then $A = 18$, $B = 18$ and $C = 9$, all divisible by 9.

4. We can think of a domino as two unit squares side-by-side. Similarly, a tetromino is four unit squares joined at their sides. Tetrominoes come in several different shapes, but for this problem, we consider only "L-tetrominoes", shaped either like the letter "L" or its mirror image, with three squares in a row and the fourth attached at the side at one end of the row. Decide whether or not it is possible to cover a 10×10 square with nonoverlapping L-tetrominoes, and prove that your answer is correct.



SOLUTION. It is not possible. To see this, we think of the 10×10 square as a grid of 100 unit squares, which we call "boxes". We color the boxes as follows. All ten boxes in the first column are black; all ten boxes in the second column are white; all ten boxes in the third column are black, and so on, alternating black and white. Now imagine setting one L-tetromino down onto the grid so that each of the four squares making up the tetromino covers exactly one box. There are several ways to do this, four with the long arm of the "L" vertical, and four with the long arm horizontal. A bit of experimentation shows that for all eight possibilities, there are either 3 black boxes and 1 white box covered, or else there are 3 white boxes and 1 black box covered. We define the *excess* of the tetromino to be the number of black boxes covered minus the number of white boxes covered, so we see that the excess must be ± 2.

Now suppose that the colored grid can be covered by L-tetrominoes, and note that since each has area 4 and the whole grid has area 100, we must use exactly 25 tetrominoes. Each of the four squares of each tetromino covers one box of the grid, so an excess of ± 2 is defined for each of the 25 tetrominoes. Also, since the whole grid is covered and it has equal numbers of black and white boxes, the total of the 25 excesses must be 0. Since 25 is odd, the number of excesses equal to 2 and the number equal to -2 cannot be equal. If the number of $+2$ excesses exceeds the number of -2 excesses, the total will be positive, and in the opposite case, the total will be negative. In neither case therefore, will the total excess be 0, and this contradiction shows that the 10×10 square cannot be covered with L-tetrominoes.

5. Let a and b be positive integers. Find all solutions to the equation $a^b + b^a = 2ab$.

SOLUTION. Suppose first that $a = 1$. Then the equation becomes $1 + b = 2b$ and hence $b = 1$. Similarly, $b = 1$ implies that $a = 1$, and note that $a = b = 1$ is indeed a solution. We can now assume that both a and b are at least 2. Then $2ab = a^b + b^a \geq a^2 + b^2$, so $0 \geq a^2 - 2ab + b^2 = (a - b)^2$. This shows that $a - b = 0$ and hence that $a = b$. But, with $a = b$, the equation becomes $2a^a = 2a^2$, so $a > 1$ implies that $a = 2$. Again $a = b = 2$ is also seen to be a solution, so the solutions are therefore $a = b = 1$ and $a = b = 2$.