WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET V (2010-2011)

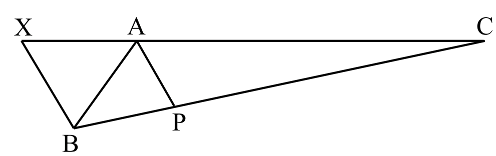
1. Find the largest positive integer n such that n is equal to the cube of the sum of its digits.

SOLUTION. Suppose that n is a k-digit number equal to the cube of its digit sum. Since each digit is at most 9, the digit sum of n is at most $9k$, so we have $(9k)^3 \geq n$. Also, since 10^{k-1} is the smallest k-digit number, we have $n \geq 10^{k-1}$, and thus $(9k)^3 \geq 10^{k-1}$. It is easy to check that this inequality holds for $k = 6$, but not for $k = 7$, and we show that the inequality fails for all $k \geq 7$. To see this, suppose that k is some integer for which the reverse inequality holds; in other words $(9k)^3 < 10^{k-1}$. If we increase k by 1, the left side increases by a factor of $((k+1)/k)^3 < 2^3 = 8$, but the right side increases by a factor of 10, so the reverse inequality continues to hold. In particular it holds for all $k \geq 7$.

It follows that $k \leq 6$. If $k = 6$, we have $n \leq 54^3 = 157464$, so the leftmost digit of n is 1 and the digit sum is at most 46. Then $n \leq 46^3 = 97336$, and hence n does not have six digits. Thus $k \leq 5$, and the digit sum s of n is at most $9k = 45$. Therefore, we need to find the largest number $s \leq 45$ such that s is the digit sum of s^3. Starting with $s = 45$ and working down (using a calculator to do the arithmetic) the first solution we find is $s = 27$, corresponding to $n = s^3 = 19683$.

The number of possibilities that must be checked can be reduced if we use the observation that for each positive integer n, the digit sum of n is congruent modulo 9 to n. Since we want $s^3 = n$, where s is the digit sum of n, this means that $s^3 - s$ must be divisible by 9, and it is easy to see that this forces s to be congruent to 0, 1 or 8 modulo 9. Thus working down from $s = 45$, we need only check 45, 44, 37, 36, 35, 28 and 27 to establish that $s = 27$ yields the largest solution.

2. In $\triangle ABC$, angle A is $120°$ and the lengths of sides $\overline{AB}$ and $\overline{AC}$ are 4 and 12, respectively. If $\overline{AP}$ bisects angle A, where P lies on $\overline{BC}$, compute the length AP (with proof, of course).



SOLUTION. Extend side $\overline{CA}$ past A to point X so that $AX = 4$. Since $AX = 4 = AB$, we see that $\triangle XAB$ is isosceles, and thus $\angle X = \angle ABX$. But $\angle XAB$ is supplementary to $\angle BAC = 120°$, so $\angle XAB = 60°$, and it follows that all three angles of $\triangle XAB$ are equal. Thus this triangle is equilateral, and in particular, $XB = 4$.

Now $\angle X = 60° = \angle PAC$, and thus $\overline{XB}$ is parallel to $\overline{AP}$. It follows that $\triangle CAP$ is similar to $\triangle CXB$, and we have $AP/XB = AC/XC$. We know, however, that $XB = 4$, $AC = 12$ and $XC = 4 + 12 = 16$. Thus $AP/4 = 12/16 = 3/4$, and we conclude that $AP = 3$.

3. The integers from 0 to 100 are written in order, horizontally across a blackboard, with spaces between the numbers. Alice puts 50 plus signs and 50 stars in some random arrangement, into the 100 spaces between the numbers. Then (interpreting "$*$" as multiplication) she computes the value of what is written, obtaining the answer a. (Of course, Alice follows the usual rule: multiplication before addition. Thus, for example, the expression $0 + 1 * 2 + 3 * 4$ has the value 14.) Bob changes all of Alice's pluses to stars and all of her stars to pluses, and then he computes the value of the new expression, obtaining the answer b. If $a + b$ is a number whose rightmost four digits are 2011, prove that at least one of Alice or Bob made an arithmetic mistake.

SOLUTION. Note that $a+b$ is an odd number. We will show, however, that if A is the true value of the expression that Alice tried to evaluate, and B is the true value of Bob's expression, then $A+B$ must be even, so it cannot be that $A=a$ and $B=b$.

After Alice puts in her 50 plusses and 50 stars, every odd integer in the range 0 to 100 is of one of four types: plus on left and star on right; star on left and plus on right; plus on both sides and finally, star on both sides. Writing e, f, g and h respectively to denote the number of odd numbers of each of these four types, we see that the total number of plusses is $e+f+2g$, and the total number of stars is $e+f+2h$. Since there are equal numbers of stars and plusses, we deduce that $g=h$, so the number of odd numbers sandwiched between two stars equals the number of odd numbers sandwiched between two plusses. In other words, the expressions that Alice and Bob tried to evaluate have equal numbers of odd numbers sandwiched between two plusses.

The quantity A is a sum of products of consecutive integers, and such a product will be odd only in the case that it is a product of just one integer and that integer is odd. Thus A is congruent modulo 2 to the number of odd numbers in the range from 0 to 100 that (in Alice's expression) are surrounded on both sides by plus signs, and similarly, B is congruent modulo 2 to the number of odd numbers in Bob's expression that are surrounded by two plusses. Since we have seen that the numbers of odd numbers surrounded on both sides by plusses are equal in Alice's and Bob's expressions, it follows that A and B are either both odd or both even, so their sum is even.

4. Let x, y and z be nonzero real numbers, and suppose that $x+y+z=3$ and $xy+yz+zx=0$. Let $p=xyz$. Prove that $-4 \le p < 0$ and find all possibilities for x, y and z such that $p=-4$.

SOLUTION. Let t be any real number. Then $(t-x)(t-y)(t-z)=t^3-3t^2-p$. In particular, taking $t=x$, we get $0=x^3-3x^2-p$, so $p=x^2(x-3)$. Since by asumption, x, y and z are nonzero, we have $p \ne 0$, and if $p>0$, we have $x^2(x-3)>0$, and thus $x>3$. Similarly $y>3$ and $z>3$. But $x+y+z=3$, so this is clearly impossible, and we deduce that $p<0$.

Now suppose $p \le -4$. Since $x+y+z=3$, at least one of x, y or z must be positive, so we can assume that $x>0$. We saw before that $x^3-3x^2-p=0$, so $x^3-3x^2=p \le -4$, and thus $x^3-3x^2+4 \le 0$. But x^3-3x^2+4 factors as $(x-2)^2(x+1)$, so we have $(x-2)^2(x+1) \le 0$. Since $x>0$, certainly $x+1>0$, and of course $(x-2)^2 \ge 0$. Thus $(x-2)^2(x+1) \ge 0$, and we conclude that $(x-2)^2(x+1)=0$. Because $x \ne -1$, we have $x=2$.

Continuing to assume that $p \le -4$, we have seen that $x=2$. Since $x+y+z=3$, at least one of y or z must be positive, and we can assume that $y>0$. Now repeating the argument of the previous paragraph we deduce that $y=2$, and thus $z=-1$. Then $xy+yz+zx=4-2-2=0$, as required, and thus $x=2$, $y=2$ and $z=-1$ is a solution, and here we have $p=-4$. The assumption that $p \le -4$, therefore, yields $p=-4$, so in all cases, $p \ge -4$, as wanted. Also, we have seen that to get $p=-4$, two of the unknowns must equal 2, and the third equals -1. Thus there are three solutions for (x,y,z) yielding $p=-4$, namely $(2,2,-1)$, $(2,-1,2)$ and $(-1,2,2)$.

5. In a certain country, a dollar is 100 cents and coins have denominations 1, 2, 5, 10, 20, 50 and 100 cents. Suppose that one can make A cents using exactly B coins. Prove that it is possible to make B dollars using exactly A coins.

SOLUTION. Let $n(d)$ be the number of d-cent coins that are used to make A cents with B coins. Then $A=1{\cdot}n(1)+2{\cdot}n(2)+5{\cdot}n(5)+\cdots+100{\cdot}n(100)$ and $B=n(1)+n(2)+n(5)+\cdots+n(100)$. The second equation yields $100B=1{\cdot}n(1)(100)+2{\cdot}n(2)(50)+5{\cdot}n(5)(20)+\cdots+100{\cdot}n(100)(1)$, and thus if we use $d{\cdot}n(d)$ coins of value $100/d$, we get $100B$ cents, which is B dollars. The total number of coins used when doing this is $1{\cdot}n(1)+\cdots+100{\cdot}n(100)=A$, as wanted.