WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

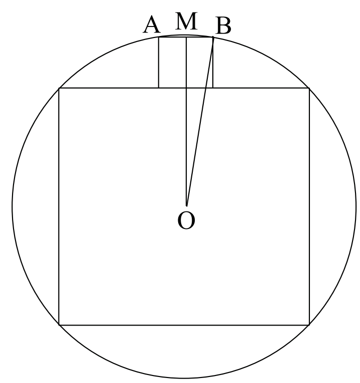
SOLUTIONS TO PROBLEM SET IV (2010-2011)

1. Suppose that for each integer x, a certain integer x^* is defined, where x^* depends on x. Assume that there are only finitely many different numbers x^* as x runs over the set of all integers, and assume also that when x and y are different integers, it is never the case that $x + kx^* = y + ky^*$, for any integer k. Show that there exists a positive integer m such that $(x + m)^* = x^*$ for all integers x.

SOLUTION. Since there are only finitely many different numbers of the form x^*, we can choose integers a and b such that $b^* - a^*$ is as large as possible, and we write $d = b^* - a^*$. Note that $d \geq 0$, and if $d = 0$, then all numbers x^* are equal, so we can take $m = 1$. Otherwise, let m be the product of all of the numbers in the set $S = \{1, 2, 3, \ldots, d-1, d\}$. (Recall that m is called "d-factorial", and that the standard notation for this is $m = d!$.)

Given an arbitrary integer x, we argue that $(x + m)^* = x^*$. Otherwise, writing $y = x + m$, we see that $x^* - y^*$ is nonzero, and so it is plus or minus one of the numbers in the set S. Then $x^* - y^*$ is a divisor of $m = y - x$, and so $(y - x)/(x^* - y^*)$ is an integer, which we call k. Then $(y - x) = k(x^* - y^*)$, so $y + ky^* = x + kx^*$. Since $y \neq x$, this is impossible by assumption, and this contradiction shows that $(x + m)^* = x^*$, as wanted.

2. In the diagram, a square is inscribed in a circle, and a smaller square is drawn with one side along a side of the large square and two vertices on the circle. What fraction of the side length of the large square is the side length of the small square?



SOLUTION. We can assume that the side length of the large square is 2 units. Then the length of a diagonal of that square is $2\sqrt{2}$, and hence the radius of the circle is $\sqrt{2}$. Let A and B be the vertices of the small square that lie on the circle, and let O be the center of the circle. Then $\overline{OB}$ is a radius, so $OB = \sqrt{2}$. Let M be the midpoint of chord $\overline{AB}$, and draw $\overline{OM}$. Then $\overline{OM}$ is perpendicular to $\overline{AB}$, and thus $\triangle OMB$ is a right triangle with hypotenuse $\overline{OB}$.

Let $2s$ be the side length of the small square, and observe that $OM = 1 + 2s$ since the distance from O to a side of the large square is half the side length of that square, which we are assuming is 2. Since $MB = s$ and $OB = \sqrt{2}$, the Pythagorean theorem yields $(1 + 2s)^2 + s^2 = (\sqrt{2})^2$, and thus $1 + 4s + 4s^2 + s^2 = 2$. Then $5s^2 + 4s - 1 = 0$. By factoring (or by using the quadratic formula) we deduce that either $s = 1/5$ or $s = -1$. Of course, only the positive solution is relevant, and we conclude that $s = 1/5$, so the side length of the small square is $2/5$, which is exactly $1/5$ of the side length of the large square.

3. Find all solutions of the simultaneous equations

$$2x^2 = 14 + yz \qquad 2y^2 = 14 + zx \qquad 2z^2 = 14 + xy\,.$$

SOLUTION. First, if $x = y = z$, then $2x^2 = 14 + x^2$, so $x^2 = 14$, and this yields two solutions for the triple (x, y, z), namely $(\sqrt{14}, \sqrt{14}, \sqrt{14})$ and $(-\sqrt{14}, -\sqrt{14}, -\sqrt{14})$. To see if there are any other solutions, we assume now that not all of the unknowns are equal.

Subtracting the second equation from the first, we get $2x^2 - 2y^2 = yz - xz$, so $2(x+y)(x-y) = z(y-x)$. If $x \neq y$, we can divide by $x-y$ to obtain $2(x+y) = -z$, or equivalently, $2x+2y+z = 0$. If also $y \neq z$, similar reasoning yields $2y+2z+x = 0$, and subtracting the second of these equations from the first, we get $x - z = 0$, so $x = z$. This shows that two of the three unknowns must be equal, and because of the symmetry of the problem, we can assume that $y = z$.

We are assuming that the three unknowns are not all equal, and thus $x \neq y$, and as we have seen, this implies that $2x + 2y + z = 0$. Since $y = z$, we have $2x + 3y = 0$, and thus $y = -(2/3)x$. Also, $2x^2 = 14 + yz = 14 + y^2 = 14 + (4/9)x^2$, and thus $(14/9)x^2 = 14$, so $x^2 = 9$ and $x = \pm 3$. If $x = 3$, then $y = -(2/3)x = -2$, and since $z = y$, this yields the solution $(3, -2, -2)$ for (x, y, z), and if $x = -3$, we get the solution $(-3, 2, 2)$. Thus in addition to the two solutions we found previously with $x = y = z$, there are exactly six other solutions: $(3, -2. -2)$, $(-2, 3, -2)$, $(-2, -2, 3)$ and the negatives of these. It is routine to check that all of these really are solutions.

4. Suppose that n is a positive integer that is not a power of 2. Show that n can be written as a sum of at least two consecutive integers.

SOLUTION. First, we recall the standard fact that if $s = 1 + 2 + 3 + \cdots + (k-1)$, then $s = k(k-1)/2$. An easy way to see this is to add the previous equation for s term by term to the reverse equation $s = (k-1) + (k-2) + \cdots + 1$. What results is that $2s$ is a sum of $k-1$ terms, each equal to k, so $2s = k(k-1)$ and $s = k(k-1)/2$, as claimed. Thus the sum of the k consecutive integers beginning with an integer m is $m + (m+1) + \cdots + (m+k-1) = km + k(k-1)/2$.

Since n is not a power of 2, we can write $n = ab$, where a is an odd number exceeding 1. First, suppose that $a < 2b$, and let $m = b - (a-1)/2$. Of course, m is an integer because a is odd, and $m > 0$ since $b > a/2 > (a-1)/2$. By the above formula, the sum of the a consecutive numbers beginning with m is $am + a(a-1)/2 = a(m + (a-1)/2) = ab = n$, and we are done in this case.

In the remaining case, $a > 2b$. Now we let $m = (a+1)/2 - b$, and we see that m is an integer because a is odd, and $m > 0$ because $(a+1)/2 > a/2 > b$. The sum of the $2b$ consecutive integers starting with m is $2bm + 2b(2b-1)/2 = b(2m + 2b - 1) = ba = n$, and we are done here also.

5. I know that the population of a certain village consists of 100 humans and 15 extraterrestrial aliens. When I visit the village, I am unable to distinguish the humans from the aliens, but I want to find someone who I am sure is human. I ask each villager to give me the names of some human residents, allowing self-nominations. Everyone submits a list of 15 names, but while the humans' lists contain only the names of humans, I have no such assurance about the aliens' lists. Show that I can select a human when I examine the 115 unsigned lists.

SOLUTION. If any name appears on 16 or more lists, then some human must have nominated that person, who is therefore human. We can thus assume that no one was nominated more than 15 times. Now, there are a total of $115 \cdot 15$ nominations, so the average number of times each resident is nominated is 15. Since no resident had an above-average number of nominations, it follows that no person had a below-average number of nominations, so each name occurs on exactly 15 lists. Each of the 15 aliens, therefore, was nominated by all of the aliens, so all aliens' lists contain the same 15 names. To find a human, I group together identical lists, and I discard all groupings of 15 identical lists. Since 115 is not a multiple of 15, there must be some lists left, and I can be sure that all names on all of those lists are the names of humans.