WISCONSIN MATHEMATICS, SCIENCE & ENGINEERING TALENT SEARCH

SOLUTIONS TO PROBLEM SET III (2010-2011)

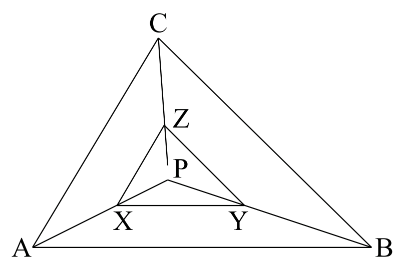
1. I have five clocks, each of which chimes when it shows a full hour. These clocks run at constant, but possibly different and incorrect rates, but nevertheless, I notice that at each hour (according to my accurate wristwatch) at least two of my clocks chime. Prove that we can throw away at least three of the five clocks and still hear a chime at each true hour.

SOLUTION. If one of the five clocks chimes at two consecutive true hours, then it chimes at every true hour (and perhaps more often). In that case, we could throw away the other four clocks, and we are done. We can thus assume that none of the clocks chimes at two consecutive true hours.

At least two clocks chime at the true 1 o'clock, so we can assume that clocks A and B chime at 1, and hence neither A nor B chimes at 2. We can thus assume that C and D chime at 2, so neither of them chimes at 3. Thus at least two of clocks A, B and E chime at 3, so one of A or B must chime at 3, and we can assume it is A. Then A chimes at 1 and 3, so it chimes at all odd true hours.

In other words, we have shown that there is an "odd clock" that chimes at all odd true hours. In the same way, starting our argument at 2 o'clock, we can show that there is an "even clock" that chimes at all even true hours. Together, these two clocks chime at all true hours, and we can therefore throw away the remaining three clocks.

2. In the figure, $\triangle XYZ$ is inside $\triangle ABC$, and sides $\overline{AB}$, $\overline{AC}$ and $\overline{BC}$ are parallel to sides $\overline{XY}$, $\overline{XZ}$ and $\overline{YZ}$, respectively. Prove that lines $\overline{AX}$, $\overline{BY}$ and $\overline{CZ}$ go through a common point.



SOLUTION. Let P be the point where line $\overline{AX}$ meets line $\overline{BY}$. First, we show that $\triangle ABC$ is similar to $\triangle XYZ$. One way to see this is to observe that $\angle BAP = \angle YXP$ since $\overline{AB}$ is parallel to $\overline{XY}$, and similarly, $\angle CAP = \angle ZXP$. Addition then yields $\angle BAC = \angle YXZ$. Similar reasoning shows that $\angle ABC = \angle XYZ$, and thus the two given triangles are indeed similar.

Since $\overline{XY}$ is parallel to $\overline{AB}$, it follows that $\triangle APB$ is similar to $\triangle XPY$, and thus we have

$$\frac{AX}{XP} + 1 = \frac{AX + XP}{XP} = \frac{AP}{XP} = \frac{AB}{XY}\,.$$

Furthermore, if we let Q be the point where line $\overline{AX}$ meets line $\overline{CZ}$, then similar reasoning yields $(AX/XQ) + 1 = AC/XZ$. Because the two original triangles are similar, we have $AB/XY = AC/XZ$, and thus

$$\frac{AX}{XP} + 1 = \frac{AB}{XY} = \frac{AC}{XZ} = \frac{AX}{XQ} + 1\,.$$

We deduce that $AX/XP = AX/XQ$, and hence $XP = XQ$. Since points P and Q lie on the same line $\overline{AX}$ containing X, and they are on the same side of X, we conclude that P and Q are really the same point.

3. Let $a < b$ be positive integers, where $a + b$ is odd, and let k be any odd positive integer. Show that the number

$$a^k + (a+1)^k + (a+2)^k + \cdots + (b-1)^k + b^k$$

is a multiple of $a + b$.

SOLUTION. The number of terms in the sum is $b - a + 1$, and this is even since $a + b$ is odd. We can thus pair the terms in the sum, with the first term paired with the last, the second term paired with the next-to-last, and so on. Thus in general, the term $(a + i)^k$ is paired with $(b - i)^k$ for $0 \le i < (b - a)/2$. Now write $n = a + b$, and observe that it suffices to show that the sum of the two terms in each pair is a multiple of n. Consider, for example, $(a + i)^k + (b - i)^k$. Writing $u = a + i$, we see that $b - i = n - a - i = n - u$, so our sum of two paired terms is $u^k + (n - u)^k$. If we expand $(n - u)^k$, we obtain a number of terms having n as a factor, and one additional term, namely $(-u)^k = -u^k$, where this equality holds since k is odd. Thus $u^k + (n - u)^k$ is a sum of terms each of which is a multiple of n, and this shows that each sum of two paired terms is a multiple of n, as wanted.

4. (New Year's Problem.) Suppose that $\square$ is an operation defined on the integers, and assume that the following conditions are satisfied.

(a) $(x + y) \square z = y \square (x + z)$ for all x, y and z.

(b) $(3x) \square y = x \square (3y)$ for all x and y.

(c) $1 \square 1 = 2011$.

Compute $2011 \square 2011$.

SOLUTION. Substitute $z = 0$ in (a) to conclude that

$$y \square x = (x + y) \square 0 = (y + x) \square 0 = x \square y$$

for all x and y. This together with (b) yields $(3x + y) \square 0 = (3x) \square y = x \square (3y) = (x + 3y) \square 0$ for all x and y. Also, we have $2 \square 0 = 1 \square 1 = 2011$ by (c) and $2011 \square 2011 = 4022 \square 0$. Next, we seek integers x and y such that $3x + y = 2$ and $x + 3y = 4022$. Solving these simultaneous equations, we find that $x = -502$ and $y = 1508$. Then, using these values of x and y, we have

$$2011 \square 2011 = 4022 \square 0 = (x + 3y) \square 0 = (3x + y) \square 0 = 2 \square 0 = 2011\,.$$

5. Prove that

$$2^{\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}\right)} < n$$

for all integers $n \ge 2$.

SOLUTION. The inequality certainly holds when $n = 2$ since $2^{1/2} = \sqrt{2} < 1.42 < 2$. We will show that if the desired inequality holds for some value of n, then it also holds for $n + 1$, and the result will follow. Assuming the inequality for n, we have

$$2^{\left(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \frac{1}{n+1}\right)} = 2^{\left(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right)} \cdot 2^{\left(\frac{1}{n+1}\right)} < n \cdot 2^{\left(\frac{1}{n+1}\right)}\,,$$

so it suffices to show that $n \cdot 2^{1/(n+1)} \le n + 1$. Equivalently, we want $2 \cdot n^{(n+1)} \le (n + 1)^{(n+1)}$.

Now if $k \ge 2$ is an integer, we can expand $(n + 1)^k$ to get

$$(n + 1)^k = n^k + k \cdot n^{k-1} + \text{positive terms} > n^k + kn^{k-1}\,.$$

In particular, if we take $k = n + 1$, this yields

$$(n + 1)^{(n+1)} > n^{(n+1)} + (n + 1)n^n = n^{(n+1)} + n^{(n+1)} + n^n > 2 \cdot n^{(n+1)}\,,$$

as wanted.